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Is Timed Branching Bisimilarity a Congruence Indeed?

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Abstract. We show that timed branching bisimilarity as defined by Van der Zwaag [17] and Baeten and Middelburg [2] is not an equivalence relation, in case of a dense time domain. We propose an adaptation based on Van der Zwaag's definition, and prove that the resulting timed branching bisimilarity is an equivalence indeed. Furthermore, we prove that in case of a discrete time domain, Van der Zwaag's definition and our adaptation coincide. Finally, we prove that a rooted version of timed branching bisimilarity is a congruence over a basic timed process algebra containing parallelism, successful termination and deadlock.

1. Introduction

Branching bisimilarity [8, 9] is a widely used concurrency semantics for process algebras that include the silent step τ. Two processes are branching bisimilar if they can be related by some branching bisimulation relation. See the work of [7] for a clear account on the strong points of branching bisimilarity.

Over the years, process algebras such as CCS, CSP and ACP have been extended with a notion of time. As a result, the concurrency semantics underlying these process algebras have been adapted to cope with the presence of time. Klusener [11, 12, 13] was the first to extend the notion of a branching

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bisimulation relation to a setting with time. The main complication is that while a process can let time pass without performing an action, such idling may mean that certain behavioural options in the future are being discarded. Klusener pioneered how this aspect of timed processes can be taken into account in a branching bisimulation context. Based on his work, Van der Zwaag [17, 18] and Baeten and Middelburg [2] proposed new notions of a timed branching bisimulation relation. A main distinction between Klusener's notion and the latter ones is that he does not allow consecutive actions to happen at the same moment in time.

A key property for a semantics is that it is an equivalence. In general, for concurrency semantics in the presence of τ, reflexivity and symmetry are easy to see, but transitivity is much more difficult. In particular, the transitivity proof for branching bisimilarity in [8] turned out to be flawed, because the relation composition of two branching bisimulation relations need not be a branching bisimulation relation. Basten [3] pointed out this flaw, and proposed a new transitivity proof for branching bisimilarity, based on the notion of a *semi*-branching bisimulation relation. Such relations are preserved under transitive closure, and the notions of branching bisimilarity and semi-branching bisimilarity coincide.

In a setting with time, proving equivalence of a concurrency semantics becomes even more complicated, compared to the untimed case. Still, equivalence properties for timed semantics are often claimed, but hardly ever proved. In [13, 17, 18, 2], equivalence properties are claimed without an explicit proof, although in all cases it is stated that such proofs do exist.

Related to this, it is an interesting question whether a rooted version of timed branching bisimilarity is a congruence over a basic timed process algebra containing parallelism, successful termination and deadlock (such as Baeten and Bergstra's $\mathrm{BPA}^{ur}_{\rho\delta}$ [1], which is a basic real time process algebra with time stamped urgent actions). Similar to equivalence, congruence properties for timed branching bisimilarity are often claimed, but hardly ever proved. One such congruence proof is provided by Klusener [13]. Considering other timed settings, Reniers and Van Weerdenburg [15] provide a congruence proof for a setting with an untimed τ-step, which makes it possible for them, unlike for us, to follow the format of the usual congruence proof for untimed branching bisimilarity. Trčka [16] proved timed branching bisimilarity to be a congruence over a timed process algebra in a setting with discrete, relative time.

In the current paper, first of all, we study in how far the notion of timed branching bisimilarity of Van der Zwaag constitutes an equivalence relation. This part was reported earlier in [6]. We give a counter-example to show that in case of a dense time domain, his notion is not transitive. We proceed to present a stronger version of Van der Zwaag's definition (stronger in the sense that it relates fewer processes), and prove that this adapted notion does constitute an equivalence relation, even when the time domain is dense. Our proof follows the approach of Basten. Next, we show that in case of a discrete time domain, Van der Zwaag's notion of timed branching bisimilarity and our new notion coincide. So in particular, in case of a discrete time domain, Van der Zwaag's notion does constitute an equivalence relation.

In Appendix B, we show that our counter-example for transitivity also applies to a notion of timed branching bisimilarity by Baeten and Middelburg in case of a dense time domain (see [2, Section 6.4.1]). So that notion does not constitute an equivalence relation either. We note that our counter-example does not apply to Klusener's version of timed branching bisimilarity, because the example uses in an essential way that consecutive actions can happen at the same moment in time.

Following the equivalence proof, we prove that a rooted version of the stronger version of timed branching bisimilarity is a congruence over a basic timed process algebra containing parallelism, successful termination and deadlock. In a number of ways, our proof differs from the usual congruence proof for untimed branching bisimilarity. Example 6.1 ($\tau(0)\cdot b(1)$ and $b(1)$ are timed branching bisimilar at time 0, but $a(1)\cdot\tau(0)\cdot b(1)$ and $a(1)\cdot b(1)$ are not) demonstrates that the standard approach for untimed branching bisimilarity, i.e. take the smallest congruence closure and prove that this yields a branching bisimulation, falls short in a timed setting. Furthermore, due to the presence of successful termination, there is an excessive number of cases. In fact, the presentation of the congruence proof for the parallel composition operator is restricted to a setting without successful termination, since the number of cases in a proof considering successful termination is just too large.

This paper is organised as follows. Section 2 contains the preliminaries, describing the notion of a *timed labelled transition system*, i.e. a timed state space. Section 3 features a counter-example to show that the notion of timed branching bisimilarity by Van der Zwaag is not an equivalence relation in case of a dense time domain. A new definition of timed branching bisimulation is proposed in Section 4, and we prove that our notion of timed branching bisimilarity is an equivalence indeed. In Section 5, we prove that in case of a discrete time domain, our definition and Van der Zwaag's definition of timed branching bisimilarity coincide. In Section 6, we prove that our definition constitutes a congruence, for a simple timed process algebra with timed actions and alternative, sequential, and parallel composition. Section 7 presents some conclusions. Appendix A contains the proofs of three lemmas for the congruence result. In Appendix B, we show that our counter-example for transitivity also applies to the notion of timed branching bisimilarity by Baeten and Middelburg [2].

2. Timed Labelled Transition Systems

Let Act be a non-empty set of visible actions, and τ a special action to represent internal events, with $\tau \notin Act$. We use Act_τ to denote $Act \cup \{\tau\}$.

The time domain $Time$ is a totally ordered set with a least element 0. We say that $Time$ is *discrete* if for each pair $u, v \in Time$ there are only finitely many $w \in Time$ such that $u < w < v$.

We use the notion of timed labelled transition systems from [17], in which labelled transitions are provided with a time stamp. A transition (s, ℓ, u, s') expresses that state s evolves into state s' by the execution of action ℓ at (absolute) time u. Such a transition is presented as $s \xrightarrow{\ell}_u s'$. It is assumed that execution of transitions does not consume any time. To keep the definition of timed labelled transition systems clean, consecutive transitions are allowed to have decreasing time stamps; in the semantics, decreasing time stamps simply give rise to an (immediate) deadlock (see Definitions 3.2 and 4.1). To express time deadlocks, the predicate $\mathcal{U}(s, u)$ denotes that state s can let time pass until time u. A special state $\surd$ represents successful termination, expressed by the predicate $\surd\downarrow$.

Definition 2.1. (Timed labelled transition system)
A *timed labelled transition system* (TLTS) [10] is a triple $(\mathcal{S}, \mathcal{T}, \mathcal{U})$, where:

1. $\mathcal{S}$ is a set of states, including a special state $\surd$, which is the only state in which the predicate $\downarrow$ holds;
2. $\mathcal{T} \subseteq \mathcal{S} \times Act_\tau \times Time \times \mathcal{S}$ is a set of transitions;
3. $\mathcal{U} \subseteq \mathcal{S} \times Time$ is a *delay relation*, which satisfies:
   - if $\mathcal{T}(s, \ell, u, r)$, then $\mathcal{U}(s, u)$;
   - if $u < v$ and $\mathcal{U}(s, v)$, then $\mathcal{U}(s, u)$.

3. Van der Zwaag's Timed Branching Bisimulation

Van Glabbeek and Weijland [9] introduced the notion of a *branching bisimulation* relation for untimed LTSs. Intuitively, a τ-transition $s \xrightarrow{\tau} s'$ is invisible if it does not lose possible behaviour (i.e., if s and s' can be related by a branching bisimulation relation). See the work of Van Glabbeek [7] for a lucid exposition on the motivations behind the definition of a branching bisimulation relation.

The reflexive transitive closure of $\xrightarrow{\tau}$ is denoted by $\Longrightarrow$.

Definition 3.1. (Branching bisimulation [9])
Assume an untimed LTS. A symmetric binary relation $B \subseteq \mathcal{S} \times \mathcal{S}$ is a *branching bisimulation* if $s\ B\ t$ implies:

1. if $s \xrightarrow{\ell} s'$, then

   i either $\ell = \tau$ and $s'\ B\ t$,

   ii or $t \Longrightarrow \hat{t} \xrightarrow{\ell} t'$ with $s\ B\ \hat{t}$ and $s'\ B\ t'$;

2. if $s\downarrow$, then $t \Longrightarrow t'\downarrow$ with $s\ B\ t'$.

Two states s and t are *branching bisimilar*, denoted by $s \leftrightarrow_b t$, if there is a branching bisimulation B with $s\ B\ t$.

Van der Zwaag [17] defined a timed version of branching bisimulation, which takes into account time stamps of transitions and ultimate delays $\mathcal{U}(s, u)$.

For $u \in \mathit{Time}$, the reflexive transitive closure of $\xrightarrow{\tau}_u$ is denoted by $\Longrightarrow_u$.

Definition 3.2. (Timed branching bisimulation [17])
Assume a TLTS $(\mathcal{S}, \mathcal{T}, \mathcal{U})$. A collection B of symmetric binary relations $B_u \subseteq \mathcal{S} \times \mathcal{S}$ for $u \in \mathit{Time}$ is a *timed branching bisimulation* if $s\ B_u\ t$ implies:

1. if $s \xrightarrow{\ell}_u s'$, then

   i either $\ell = \tau$ and $s'\ B_u\ t$,

   ii or $t \Longrightarrow_u \hat{t} \xrightarrow{\ell}_u t'$ with $s\ B_u\ \hat{t}$ and $s'\ B_u\ t'$;

2. if $s\downarrow$, then $t \Longrightarrow_u t'\downarrow$ with $s\ B_u\ t'$;

3. if $u \leq v$ and $\mathcal{U}(s, v)$, then for some $n \geq 0$ there are $t_0, \ldots, t_n \in \mathcal{S}$ with $t = t_0$ and $\mathcal{U}(t_n, v)$, and $u_0 < \cdots < u_n \in \mathit{Time}$ with $u = u_0$ and $v = u_n$, such that for $i < n$, $t_i \Longrightarrow_{u_i} t_{i+1}$, $s\ B_{u_i}\ t_{i+1}$ and $s\ B_{u_{i+1}}\ t_{i+1}$.

Two states s and t are *timed branching bisimilar at* u, denoted by $s \leftrightarrow_{tb}^{Z,u} t$, if there is a timed branching bisimulation B with $s\ B_u\ t$. States s and t are *timed branching bisimilar*, denoted by $s \leftrightarrow_{tb}^{Z} t$,[1] if they are timed branching bisimilar at all $u \in \mathit{Time}$.

[1] The superscript Z refers to van der Zwaag, to distinguish it from the adaptation of his definition of timed branching bisimulation that we will define later.

Transitions can be executed at the same time consecutively. By the first clause in Definition 3.2, the behaviour of a state at some point in time is treated like untimed behaviour. The second clause deals with successful termination.[2] By the last clause, time passing in a state s is matched by a related state t with a "τ-path" where all intermediate states are related to s at times when a τ-transition is performed.[3]

In the following examples, $\mathbb{Z}_{\geq 0} \subseteq Time$, and, for any states $s_0, s_1 \in \mathcal{S}$, if $s_0 \xrightarrow{\ell}_u s_1$, then $\mathcal{U}(s_0, u)$ and for all $v > u$, $\neg\mathcal{U}(s_0, v)$.

Example 3.1. Consider the following two TLTSs: $s_0 \xrightarrow{a}_2 s_1 \xrightarrow{b}_1 s_2$, $t_0 \xrightarrow{a}_2 t_1$, and $\mathcal{U}(t_1, 1)$. We have $s_0 \leftrightarrow^Z_{tb} t_0$, since $s_0 \ B_w \ t_0$ for $w \geq 0$, $s_1 \ B_w \ t_1$ for $w > 1$, and $s_2 \ B_w \ t_1$ for $w \geq 0$ is a timed branching bisimulation.

Example 3.2. Consider the following two TLTSs: $s_0 \xrightarrow{a}_1 s_1 \xrightarrow{\tau}_2 s_2 \xrightarrow{b}_3 s_3$, $t_0 \xrightarrow{a}_1 t_1 \xrightarrow{b}_3 t_2$, $\mathcal{U}(s_3, 4)$, and $\mathcal{U}(t_2, 4)$. We have $s_0 \leftrightarrow^Z_{tb} t_0$, since $s_0 \ B_w \ t_0$ for $w \geq 0$, $s_1 \ B_w \ t_1$ for $w \leq 2$, $s_2 \ B_w \ t_1$ for $w \geq 0$, and $s_3 \ B_w \ t_2$ for $w \geq 0$ is a timed branching bisimulation.

Example 3.3. Consider the following two TLTSs: $s_0 \xrightarrow{a}_u s_1 \xrightarrow{\tau}_v s_2 \downarrow$ and $t_0 \xrightarrow{a}_u t_1 \downarrow$. If $u = v$, we have $s_0 \leftrightarrow^Z_{tb} t_0$, since $s_0 \ B_w \ t_0$ for $w \geq 0$, $s_1 \ B_u \ t_1$, and $s_2 \ B_w \ t_1$ for $w \geq 0$ is a timed branching bisimulation. If $u \neq v$, we have $s_0 \not\leftrightarrow^Z_{tb} t_0$, because s_1 and t_1 are not timed branching bisimilar at time u; namely, t_1 has a successful termination, and s_1 cannot simulate this at time u, as it cannot do a τ-transition at time u.

Example 3.4. Consider the following two TLTSs: $s_0 \xrightarrow{\tau}_u s_1 \xrightarrow{a}_v s_2 \downarrow$ and $t_0 \xrightarrow{a}_v t_1 \downarrow$. If $u = v$, we have $s_0 \leftrightarrow^Z_{tb} t_0$, since $s_0 \ B_w \ t_0$ for $w \geq 0$, $s_1 \ B_w \ t_0$ for $w \geq 0$, and $s_2 \ B_w \ t_1$ for $w \geq 0$ is a timed branching bisimulation. If $u \neq v$, we have $s_0 \not\leftrightarrow^Z_{tb} t_0$, because s_0 and t_0 are not timed branching bisimilar at time $\frac{u+v}{2}$.[4]

Van der Zwaag [17, 18] wrote about his definition: "It is straightforward to verify that branching bisimilarity is an equivalence relation." However, we found that in general this is not the case. A counter-example is presented below. Note that it uses a dense time domain.

Example 3.5. Let p, q, and r defined as in Figures 1, 2 and 3, with $Time = \mathbb{Q}_{\geq 0}$. We depict $s \xrightarrow{a}_u s'$ as $s \xrightarrow{a(u)} s'$.

$p \leftrightarrow^Z_{tb} q$, since $p \ B_w \ q$ for $w \geq 0$, $p_i \ B_w \ q_i$ for $w \leq \frac{1}{i+2}$, and $p'_i \ B_w \ q_i$ for $w > 0$ (for $i \geq 0$) is a timed branching bisimulation.

Moreover, $q \leftrightarrow^Z_{tb} r$, since $q \ B_w \ r$ for $w \geq 0$, $q_i \ B_w \ r_i$ for $w \geq 0$, $q_i \ B_0 \ r_j$, and $q_i \ B_w \ r_\infty$ for $w = 0 \vee w > \frac{1}{i+2}$ (for $i, j \geq 0$) is a timed branching bisimulation. (Note that q_i and r_∞ are *not* timed branching bisimilar in the time interval $\langle 0, \frac{1}{i+2}]$.)

[2] Van der Zwaag does not take into account successful termination, so the second clause is missing in his definition.

[3] In the definition of Van der Zwaag, instead of $u \leq v$ and $n \geq 0$, $u < v$ and $n > 0$ are written, respectively. The change is needed in order to deal correctly with the deadlock process $\delta(u)$ and the parallel composition operator $||$ later on, when we come to the congruence proof in Section 6. According to the old definition, $\delta(1) \leftrightarrow^{Z,2}_{tb} \delta(2)$, but then, since $a(2) \,||\, \delta(1) \not\leftrightarrow^{Z,2}_{tb} a(2) \,||\, \delta(2)$, the congruence proof would be broken. Instead, it is desirable that $\delta(1) \not\leftrightarrow^{Z,2}_{tb} \delta(2)$. Van der Zwaag did not consider deadlock explicitly; in the absence of deadlock, the two definitions (with '$u < v$' and '$u \leq v$') coincide.

[4] $s_0 \leftrightarrow^Z_{tb} t_0$ would hold for $u \leq v$ if in Definition 3.2 we would require that they are timed branching bisimilar at 0 (instead of at all $u \in Time$).

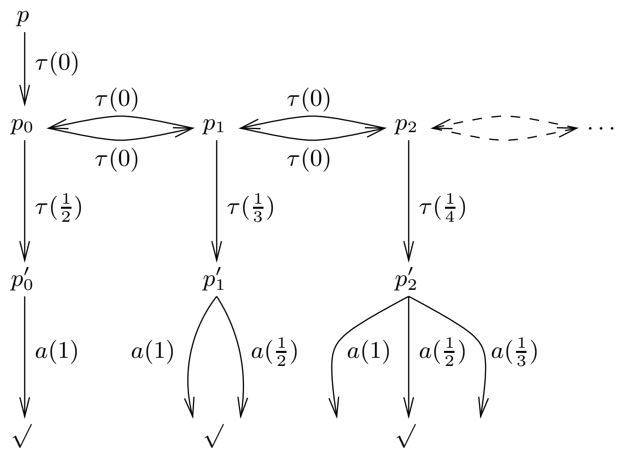

Figure 1. A timed process p

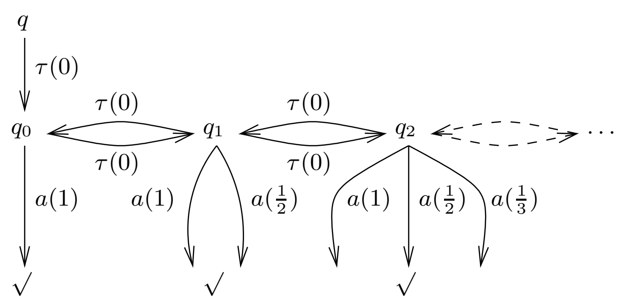

Figure 2. A timed process q

However, $p \not\leftrightarrow^{Z}_{tb} r$, due to the fact that none of the p_i can simulate r_∞. Namely, r_∞ can idle until time 1; p_i can only simulate this by executing a τ at time $\frac{1}{i+2}$, but the resulting process $\sum_{n=1}^{i+1} a(\frac{1}{n})$ is not timed branching bisimilar to r_∞ at time $\frac{1}{i+2}$, since only the latter can execute action a at time $\frac{1}{i+2}$.

4. A Strengthened Timed Branching Bisimulation

In this section, we propose a way to fix the definition of Van der Zwaag (see Definition 3.2). Our adaptation requires the *stuttering property* [9] (see Definition 4.3) at all time intervals. That is, in the last clause of Definition 3.2, we require that $s\ B_w\ t_{i+1}$ for $u_i \leq w \leq u_{i+1}$. Hence, we achieve a stronger version of Van der Zwaag's definition. We prove that this new notion of timed branching bisimilarity is an equivalence relation.

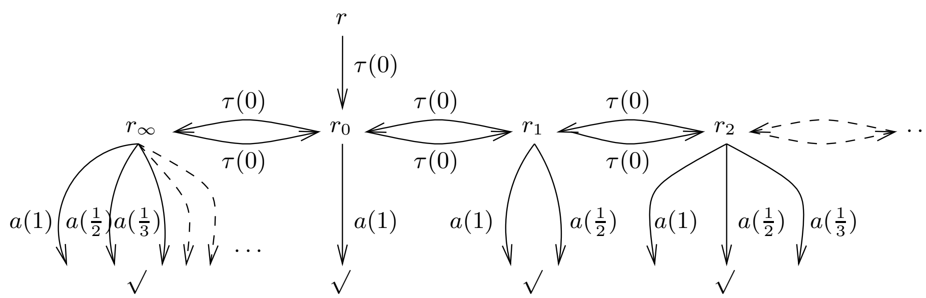

Figure 3. A timed process r

4.1. Timed Branching Bisimulation

Definition 4.1. (Timed branching bisimulation)
Assume a TLTS $(\mathcal{S}, \mathcal{T}, \mathcal{U})$. A collection B of binary relations $B_u \subseteq \mathcal{S} \times \mathcal{S}$ for $u \in \mathit{Time}$ is a *timed branching bisimulation* if $s\ B_u\ t$ implies:

1. if $s \xrightarrow{\ell}_u s'$, then

   i either $\ell = \tau$ and $s'\ B_u\ t$,

   ii or $t \Rightarrow_u \hat{t} \xrightarrow{\ell}_u t'$ with $s\ B_u\ \hat{t}$ and $s'\ B_u\ t'$;

2. if $t \xrightarrow{\ell}_u t'$, then vice versa;

3. if $s \downarrow$, then $t \Rightarrow_u t' \downarrow$ with $s\ B_u\ t'$;

4. if $t \downarrow$, then vice versa;

5. if $u \leq v$ and $\mathcal{U}(s, v)$, then for some $n \geq 0$ there are $t_0, \ldots, t_n \in \mathcal{S}$ with $t = t_0$ and $\mathcal{U}(t_n, v)$, and $u_0 < \cdots < u_n \in \mathit{Time}$ with $u = u_0$ and $v = u_n$, such that for $i < n$, $t_i \Rightarrow_{u_i} t_{i+1}$ and $s\ B_w\ t_{i+1}$ for $u_i \leq w \leq u_{i+1}$;

6. if $u \leq v$ and $\mathcal{U}(t, v)$, then vice versa.

Two states s and t are *timed branching bisimilar at* u, denoted by $s \leftrightarrow_{tb}^{u} t$, if there is a timed branching bisimulation B with $s\ B_u\ t$. States s and t are *timed branching bisimilar*, denoted by $s \leftrightarrow_{tb} t$, if they are timed branching bisimilar at all $u \in \mathit{Time}$.

It is not hard to see that the union of timed branching bisimulations is again a timed branching bisimulation.

The difference between Definitions 3.2 and 4.1 lies in the stuttering property. In clauses 5 and 6 of Definition 4.1, in addition to the requirement that time passing in a state s is matched by a related state t with a "τ-path" where all intermediate states are related to s at times when a τ-transition is performed, all intermediate states also need to be related to s between these times. Note that states q and r from Example 3.5 are not timed branching bisimilar according to Definition 4.1. Namely, none of the q_i can simulate r_∞ in the time interval $\langle 0, \frac{1}{i+2}]$, so that the stuttering property is violated.

Starting from this point, we focus on timed branching bisimulation as defined in Definition 4.1. We did not define this new notion of timed branching bisimulation as a symmetric relation (like in Definition 3.2), in view of the equivalence proof that we are going to present. Namely, in general the relation composition of two symmetric relations is not symmetric. Clearly any symmetric timed branching bisimulation is a timed branching bisimulation. Furthermore, it follows from Definition 4.1 that the inverse of a timed branching bisimulation is again a timed branching bisimulation, so the union of a timed branching bisimulation and its inverse is a symmetric timed branching bisimulation. Hence, Definition 4.1 and the definition of timed branching bisimulation as a symmetric relation give rise to the same notion.

Example 4.1. Consider the following two TLTSs: $s_0 \xrightarrow{a}_1 s_1$ and $t_0 \xrightarrow{a}_1 t_1$, with $\mathcal{U}(s_1, 0)$ and $\mathcal{U}(t_1, 1)$ We have $s_0 \not\leftrightarrow_{tb} t_0$, because s_1 and t_1 are not timed branching bisimilar at time 1; namely, t_1 can delay until time 1, and s_1 can neither delay until time 1, nor simulate this by doing τ-transitions at time 1 to a state which can delay until time 1. (Note that s_0 and t_0 are timed branching bisimilar according to the original definition of Van der Zwaag; see footnote 3).

4.2. Timed Semi-branching Bisimulation

Basten [3] showed that the relation composition of two (untimed) branching bisimulations is not necessarily again a branching bisimulation. Figure 4 illustrates an example, showing that the relation composition of two timed branching bisimulations is not always a timed branching bisimulation. It is a slightly simplified version of an example from [3], here applied at time 0. Clearly, B and D are timed branching bisimulations. However, $B{\circ}D$ is not, and the problem arises at the transition $r_0 \xrightarrow{\tau}_0 r_1$. According to case 1 of Definition 3.2, since $r_0\,(B{\circ}D)\,t_0$, either $r_1\,(B{\circ}D)\,t_0$, or $r_0\,(B{\circ}D)\,t_1$ and $r_1\,(B{\circ}D)\,t_2$, must hold. But neither of these cases hold, so $B{\circ}D$ is not a timed branching bisimulation.

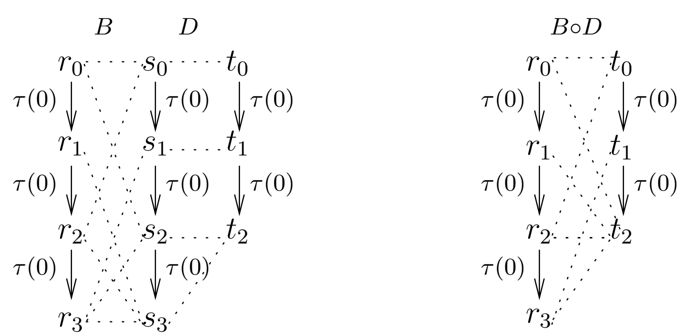

Figure 4. Composition does not preserve timed branching bisimulation

Semi-branching bisimulation [9] relaxes case 1i of Definition 3.1: if $s \xrightarrow{\tau} s'$, then it is allowed that $t \Longrightarrow t'$ with $s\ B\ t'$ and $s'\ B\ t'$. Basten proved that the relation composition of two semi-branching bisimulations is again a semi-branching bisimulation. It is easy to see that semi-branching bisimilarity is reflexive and symmetric. Hence, semi-branching bisimilarity is an equivalence relation. Then he proved that semi-branching bisimilarity and branching bisimilarity coincide, that means two states in an (untimed) LTS are related by a branching bisimulation relation if and only if they are related by a semi-branching bisimulation relation. We mimic the approach of [3] to prove that timed branching bisimilarity is an equivalence relation.

Definition 4.2. (Timed semi-branching bisimulation)
Assume a TLTS $(\mathcal{S}, \mathcal{T}, \mathcal{U})$. A collection B of binary relations $B_u \subseteq \mathcal{S} \times Time \times \mathcal{S}$ for $u \in Time$ is a *timed semi-branching bisimulation* if $s\ B_u\ t$ implies:

1. if $s \xrightarrow{\ell}_u s'$, then

   i either $\ell = \tau$ and $t \Rightarrow_u t'$ with $s\ B_u\ t'$ and $s'\ B_u\ t'$,

   ii or $t \Rightarrow_u \hat{t} \xrightarrow{\ell}_u t'$ with $s\ B_u\ \hat{t}$ and $s'\ B_u\ t'$;

2. if $t \xrightarrow{\ell}_u t'$, then vice versa.

3. if $s \downarrow$, then $t \Rightarrow_u t' \downarrow$ with $s\ B_u\ t'$;

4. if $t \downarrow$, then vice versa.

5. if $u \leq v$ and $\mathcal{U}(s, v)$, then for some $n \geq 0$ there are $t_0, \ldots, t_n \in \mathcal{S}$ with $t = t_0$ and $\mathcal{U}(t_n, v)$, and $u_0 < \cdots < u_n \in Time$ with $u = u_0$ and $v = u_n$, such that for $i < n$, $t_i \Rightarrow_{u_i} t_{i+1}$ and $s\ B_w\ t_{i+1}$ for $u_i \leq w \leq u_{i+1}$;

6. if $u \leq v$ and $\mathcal{U}(t, v)$, then vice versa.

Two states s and t are *timed semi-branching bisimilar at* u if there is a timed semi-branching bisimulation B with $s\ B_u\ t$. States s and t are *timed semi-branching bisimilar* if they are timed semi-branching bisimilar at all $u \in Time$.

It is not hard to see that the union of timed semi-branching bisimulations is again a timed semi-branching bisimulation. Furthermore, any timed branching bisimulation is a timed semi-branching bisimulation.

Definition 4.3. (Stuttering property [9])
A timed semi-branching bisimulation B is said to satisfy the *stuttering property* if:

1. $s\ B_u\ t$, $s'\ B_u\ t$ and $s \xrightarrow{\tau}_u s_1 \xrightarrow{\tau}_u \cdots \xrightarrow{\tau}_u s_n \xrightarrow{\tau}_u s'$ implies that $s_i\ B_u\ t$ for $1 \leq i \leq n$;

2. $s\ B_u\ t$, $s\ B_u\ t'$ and $t \xrightarrow{\tau}_u t_1 \xrightarrow{\tau}_u \cdots \xrightarrow{\tau}_u t_n \xrightarrow{\tau}_u t'$ implies that $s\ B_u\ t_i$ for $1 \leq i \leq n$.

The following lemma for timed semi-branching bisimulations is easy to prove, in a similar fashion as the untimed case (see [3, Corollary 10]).

Lemma 4.1. Any timed semi-branching bisimulation satisfying the stuttering property is a timed branching bisimulation.

4.3. Timed Branching Bisimilarity is an Equivalence

Following [3], our equivalence proof consists of the following main steps:

1. We first prove that the relation composition of two timed semi-branching bisimulation relations is again a semi-branching bisimulation relation (Proposition 4.1).

2. Then we prove that timed semi-branching bisimilarity is an equivalence relation (Corollary 4.1).

3. Finally, we prove that the largest timed semi-branching bisimulation satisfies the stuttering property (Proposition 4.2).

According to Lemma 4.1, any timed semi-branching bisimulation satisfying the stuttering property is a timed branching bisimulation. So by the 3rd point, two states are related by a timed branching bisimulation if and only if they are related by a timed semi-branching bisimulation.

The following lemma for timed semi-branching bisimulations can be proved in the same way as in the untimed case; see [3, Lemma 6].

Lemma 4.2. Let B be a timed semi-branching bisimulation, and $s\ B_u\ t$.

1. $s \Rightarrow_u s' \implies (\exists t' \in \mathcal{S} : t \Rightarrow_u t' \wedge s'\ B_u\ t')$;

2. $t \Rightarrow_u t' \implies (\exists s' \in \mathcal{S} : s \Rightarrow_u s' \wedge t'\ B_u\ s')$.

Proposition 4.1. The relation composition of two timed semi-branching bisimulations is again a timed semi-branching bisimulation.

Proof:
Let B and D be timed semi-branching bisimulations. We prove that the composition of B and D (or better, the compositions of B_u and D_u for $u \in \mathit{Time}$) is a timed semi-branching bisimulation. Suppose that $r\ B_u\ s\ D_u\ t$ for $r, s, t \in \mathcal{S}$. We need to check that the conditions of Definition 4.2 are satisfied with respect to the pair r, t. The first three cases are identical to the proof in the untimed case; see [3, Proposition 7]. We now consider case 4.

$u \leq v$ and $\mathcal{U}(r, v)$. Since $r\ B_u\ s$, for some $n \geq 0$ there are $s_0, \ldots, s_n \in \mathcal{S}$ with $s = s_0$ and $\mathcal{U}(s_n, v)$, and $u_0 < \cdots < u_n \in \mathit{Time}$ with $u = u_0$ and $v = u_n$, such that $s_i \Rightarrow_{u_i} s_{i+1}$ and $r\ B_w\ s_{i+1}$ for $u_i \leq w \leq u_{i+1}$ and $i < n$.

For $i \leq n$ we show that for some $m_i \geq 0$ there are $t^i_0, \ldots, t^i_{m_i} \in \mathcal{S}$ with $t = t^0_0$ and $\mathcal{U}(t^n_{m_n}, v)$, and $v^i_0 \leq \cdots \leq v^i_{m_i} \in \mathit{Time}$ with (A$_i$) $u_{i-1} = v^i_0$ (if $i > 0$) and (B$_i$) $u_i = v^i_{m_i}$, such that:

(C$_i$) $t^i_j \Rightarrow_{v^i_j} t^i_{j+1}$ for $j < m_i$;

(D$_i$) $t^{i-1}_{m_{i-1}} \Rightarrow_{u_{i-1}} t^i_0$ (if $i > 0$);

(E$_i$) $s_i\ D_{u_{i-1}}\ t^i_0$ (if $i > 0$);

(F$_i$) $s_i\ D_w\ t^i_{j+1}$ for $v^i_j \leq w \leq v^i_{j+1}$ and $j < m_i$.

We apply induction with respect to i.

- *Base case:* $i = 0$.

  Let $m_0 = 0$, $t_0^0 = t$ and $v_0^0 = u_0$. Note that B_0, C_0 and F_0 hold.

- *Inductive case:* $0 < i \leq n$.

  Suppose that $m_k, t_0^k, \ldots, t_{m_k}^k, v_0^k, \ldots, v_{m_k}^k$ have been defined for $0 \leq k < i$. Moreover, suppose that B_k, C_k and F_k hold for $0 \leq k < i$, and that A_k, D_k and E_k hold for $0 < k < i$.

  F_{i-1} for $j = m_{i-1} - 1$ together with B_{i-1} yields $s_{i-1}\ D_{u_{i-1}}\ t_{m_{i-1}}^{i-1}$. Since $s_{i-1} \Rightarrow_{u_{i-1}} s_i$, Lemma 4.2 implies that $t_{m_{i-1}}^{i-1} \Rightarrow_{u_{i-1}} t'$ with $s_i\ D_{u_{i-1}}\ t'$. We define $t_0^i = t'$ [then D_i and E_i hold] and $v_0^i = u_{i-1}$ [then A_i holds]. $s_i \Rightarrow_{u_i} \cdots \Rightarrow_{u_{n-1}} s_n$ with $\mathcal{U}(s_n, v)$ implies that $\mathcal{U}(s_i, u_i)$. Since $s_i\ D_{u_{i-1}}\ t_0^i$, according to case 5 of Definition 4.2, for some $m_i > 0$ there are $t_1^i, \ldots, t_{m_i}^i \in \mathcal{S}$ with $\mathcal{U}(t_{m_i}^i, u_i)$, and $v_1^i < \cdots < v_{m_i}^i \in \mathit{Time}$ with $v_0^i < v_1^i$ and $u_i = v_{m_i}^i$ [then B_i holds], such that for $j < m_i$, $t_j^i \Rightarrow_{v_j^i} t_{j+1}^i$ [then C_i holds] and $s_i\ D_w\ t_{j+1}^i$ for $v_j^i \leq w \leq v_{j+1}^i$ [then F_i holds].

Concluding, for $i < n$, $r\ B_{u_i}\ s_{i+1}\ D_{u_i}\ t_0^{i+1}$ and $r\ B_w\ s_{i+1}\ D_w\ t_{j+1}^{i+1}$ for $v_j^{i+1} \leq w \leq v_{j+1}^{i+1}$ and $j < m_i$. Since $v_j^i \leq v_{j+1}^i$, $v_{m_i}^i = u_i = v_0^{i+1}$, $t = t_0^0$, $u = u_0 = v_0^0$, $t_j^i \Rightarrow_{v_j^i} t_{j+1}^i$, $t_{m_i}^i \Rightarrow_{u_i} t_0^{i+1}$, and $\mathcal{U}(t_{m_n}^n, v)$, we are done.

Concluding, case 5 of Definition 4.2 is satisfied. Similarly it can be checked that case 6 is satisfied. And we already remarked that cases 1-4 of Definition 4.2 are also satisfied, which can be proved in a similar fashion as in the untimed case ([3, Proposition 7]). So the composition of B and D is again a timed semi-branching bisimulation. □

Obviously, timed semi-branching bisimilarity is reflexive and symmetric, and by Proposition 4.1 it is transitive. So it constitutes an equivalence relation.

Corollary 4.1. Timed semi-branching bisimilarity is an equivalence relation.

Proposition 4.2. The largest timed semi-branching bisimulation satisfies the stuttering property.

Proof:
Let B be the largest timed semi-branching bisimulation on $\mathcal{S}$. Let $s \xrightarrow{\tau}_u s_1 \xrightarrow{\tau}_u \cdots \xrightarrow{\tau}_u s_n \xrightarrow{\tau}_u s'$ with $s\ B_u\ t$ and $s'\ B_u\ t$. We prove that $B' = B \cup \{(s_i, t) \mid 1 \leq i \leq n\}$ is a timed semi-branching bisimulation.

We need to check that all cases of Definition 4.2 are satisfied for the relations $s_i B'_u t$, for $1 \leq i \leq n$. The cases 1-4 can be dealt with as in the untimed case; see [9, Claim 2.7]. We therefore only consider the cases 5, 6.

Let $u \leq v$ and $\mathcal{U}(s_i, v)$. Since $s \Rightarrow_u s_i$ and $s\ B_u\ t$, by Lemma 4.2 $t \Rightarrow_u t'$ with $s_i\ B_u\ t'$. It follows that for some $n \geq 0$ there are $t_0, \ldots, t_n \in \mathcal{S}$ with $t' = t_0$ and $\mathcal{U}(t_n, v)$, and $u_0 < \cdots < u_n \in \mathit{Time}$ with $u = u_0$ and $v = u_n$, such that for $j < n$, $t_j \Rightarrow_{u_j} t_{j+1}$ and $s_i\ B_w\ t_{j+1}$ for $u_j \leq w \leq u_{j+1}$. Since $t \Rightarrow_u t' \Rightarrow_u t_1$, this agrees with case 5 of Definition 4.2.

Hence, case 5 of Definition 4.2 is satisfied, and in a similar fashion we can show that case 6 is also satisfied. Concluding, B' is a timed semi-branching bisimulation. Since B is the largest, and $B \subseteq B'$, we find that $B = B'$. So B satisfies the first requirement of Definition 4.3.

Since B is the largest timed semi-branching bisimulation and timed semi-branching bisimilarity is an equivalence, B is symmetric. Then B also satisfies the second requirement of Definition 4.3. Hence B satisfies the stuttering property. □

As a consequence, the largest timed semi-branching bisimulation is a timed branching bisimulation (by Lemma 4.1 and Proposition 4.2). Since any timed branching bisimulation is a timed semi-branching bisimulation, we have the following two corollaries.

Corollary 4.2. Two states are related by a timed branching bisimulation if and only if they are related by a timed semi-branching bisimulation.

Corollary 4.3. Timed branching bisimilarity, $\leftrightarrow_{tb}$, is an equivalence relation.

We note that for each $u \in \mathit{Time}$, timed branching bisimilarity at time u is also an equivalence relation.

5. Discrete Time Domains

Theorem 5.1. In case of a discrete time domain, $\leftrightarrow^Z_{tb}$ and $\leftrightarrow_{tb}$ coincide.

Proof:
Clearly $\leftrightarrow_{tb} \subseteq \leftrightarrow^Z_{tb}$. We prove that $\leftrightarrow^Z_{tb} \subseteq \leftrightarrow_{tb}$. Suppose B is a timed branching bisimulation relation according to Definition 3.2. We show that B is a timed branching bisimulation relation according to Definition 4.1. B satisfies cases 1-4 of Definition 4.1, since they coincide with cases 1-2 of Definition 3.2. We prove that case 5 of Definition 4.1 is satisfied.

Let $s\ B_u\ t$ and $\mathcal{U}(s, v)$ with $u \leq v$. Let $u_0 < \cdots < u_n \in \mathit{Time}$ with $u_0 = u$ and $u_n = v$, where $u_1, \ldots, u_{n-1}$ are all the elements from Time that are between u and v. (Here we use that Time is discrete.) We prove by induction on n that there are $t_0, \ldots, t_n \in \mathcal{S}$ with $t = t_0$ and $\mathcal{U}(t_n, v)$, such that for $i < n$, $t_i \Longrightarrow_{u_i} t_{i+1}$ and $s\ B_w\ t_{i+1}$ for $u_i \leq w \leq u_{i+1}$.

- *Base case:* $n = 0$. Then $u = v$. By case 3 of Definition 3.2, $\mathcal{U}(t, u)$.

- *Inductive case:* $n > 0$. Since $\mathcal{U}(s, v)$, clearly also $\mathcal{U}(s, u_1)$. By case 3 of Definition 3.2 there is a $t_1 \in \mathcal{S}$ such that $t \Longrightarrow_u t_1$, $s\ B_u\ t_1$ and $s\ B_{u_1}\ t_1$. Hence, $s\ B_w\ t_1$ for $u \leq w \leq u_1$. By induction, $s\ B_{u_1}\ t_1$ together with $\mathcal{U}(s, v)$ implies that there are $t_2, \ldots, t_n \in \mathcal{S}$ with $\mathcal{U}(t_n, v)$, such that for $1 \leq i < n$, $t_i \Longrightarrow_{u_i} t_{i+1}$, $s\ B_{u_i}\ t_{i+1}$ and $s\ B_{u_{i+1}}\ t_{i+1}$. Hence, $s\ B_w\ t_{i+1}$ for $u_i \leq w \leq u_{i+1}$.

We conclude that case 5 of Definition 4.1 holds. Similarly, it can be proved that B satisfies case 6 of Definition 4.1. Hence, B is a timed branching bisimulation relation according to Definition 4.1. So $\leftrightarrow^Z_{tb} \subseteq \leftrightarrow_{tb}$. □

6. Rooted Timed Branching Bisimilarity as a Congruence

6.1. Rooted Timed Branching Bisimilarity

In this section, we prove that a rooted version of the timed branching bisimulation as defined in Definition 4.1 is a congruence over a given basic process algebra with sequential, alternative, and parallel composition. Like (untimed) branching bisimilarity, timed branching bisimilarity is not a congruence over most process algebras from the literature. A rootedness condition has been introduced for branching bisimilarity to remedy this imperfection [4, 14]. First, we provide a related definition of rooted timed branching bisimulation in Definition 6.1. Following, we introduce the transition rules of a basic process algebra, encompassing atomic actions, including τ and δ, and the alternative, sequential, and parallel composition process operators. After that, the congruence proof is presented.

Definition 6.1. (Rooted timed branching bisimulation)
Assume a TLTS $(\mathcal{S}, \mathcal{T}, \mathcal{U})$. A binary relation $B \subseteq \mathcal{S} \times \mathcal{S}$ is a *rooted timed branching bisimulation* if $s\ B\ t$ implies:

1. if $s \xrightarrow{\ell}_u s'$, then $t \xrightarrow{\ell}_u t'$ with $s' \leftrightarrows_{tb}^{u} t'$;
2. if $t \xrightarrow{\ell}_u t'$, then $s \xrightarrow{\ell}_u s'$ with $s' \leftrightarrows_{tb}^{u} t'$;
3. $s \downarrow$ iff $t \downarrow$;
4. $\mathcal{U}(s, u)$ iff $\mathcal{U}(t, u)$.

Two states s and t are *rooted timed branching bisimilar*, denoted by $s \leftrightarrows_{rtb} t$, if there is a rooted timed branching bisimulation B with $s\ B\ t$.

Note that $\leftrightarrows_{rtb} \subseteq \leftrightarrows_{tb}$. A rooted timed branching bisimulation relation is a timed branching bisimulation relation, where in cases 1 to 4 of Definition 4.1 '$\Longrightarrow_u$' constitutes zero τ-steps, and in cases 5 and 6 $n = 0$.

6.2. A Basic Process Algebra

In the following, x, y are variables, p, q, r are process terms, and s, t are process terms or $\surd$, with $\surd$ a special state representing successful termination.

Here, we present a basic process algebra, which we will use in subsequent sections in our congruence proof. It is based on the process algebra BPA$_{\rho\delta U}$ [1]. (For the sake of simplicity, the integration operator, which allows alternative composition over a possibly infinite range of time elements, is not taken into account here.) We consider the following transition rules for the process algebra used, where the synchronisation of two actions a and b resulting in an action c is denoted by $a \mid b = c$. Whenever two actions a and b should never synchronise, we define that $a \mid b = \delta$.

$$\text{Termination}: \frac{}{\surd \downarrow} \qquad \text{Atomic}: \frac{}{a(u) \xrightarrow{a}_u \surd}$$

$$\text{Alt1}: \frac{x \xrightarrow{a}_u x'}{x + y \xrightarrow{a}_u x'} \qquad \text{Alt2}: \frac{x \xrightarrow{a}_u \surd}{x + y \xrightarrow{a}_u \surd}$$

$$\text{Alt3}: \frac{y \xrightarrow{a}_u y'}{x + y \xrightarrow{a}_u y'} \qquad \text{Alt4}: \frac{y \xrightarrow{a}_u \surd}{x + y \xrightarrow{a}_u \surd}$$

$$\text{Seq1}: \frac{x \xrightarrow{a}_u x'}{x \cdot y \xrightarrow{a}_u x' \cdot y} \qquad \text{Seq2}: \frac{x \xrightarrow{a}_u \surd}{x \cdot y \xrightarrow{a}_u y}$$

$$\text{Par1}: \frac{x \xrightarrow{a}_u x' \quad \mathcal{U}(y,u)}{x \,||\, y \xrightarrow{a}_u x' \,||\, y} \qquad \text{Par2}: \frac{y \xrightarrow{a}_u y' \quad \mathcal{U}(x,u)}{x \,||\, y \xrightarrow{a}_u x \,||\, y'}$$

$$\text{Par3}: \frac{x \xrightarrow{a}_u \surd \quad \mathcal{U}(y,u)}{x \,||\, y \xrightarrow{a}_u y} \qquad \text{Par4}: \frac{y \xrightarrow{a}_u \surd \quad \mathcal{U}(x,u)}{x \,||\, y \xrightarrow{a}_u x}$$

$$\text{Par5}: \frac{x \xrightarrow{a}_u x' \quad y \xrightarrow{b}_u y' \quad a \mid b = c \quad c \neq \delta}{x \,||\, y \xrightarrow{c}_u x' \,||\, y'}$$

$$\text{Par6}: \frac{x \xrightarrow{a}_u \surd \quad y \xrightarrow{b}_u y' \quad a \mid b = c \quad c \neq \delta}{x \,||\, y \xrightarrow{c}_u y'}$$

$$\text{Par7}: \frac{x \xrightarrow{a}_u x' \quad y \xrightarrow{b}_u \surd \quad a \mid b = c \quad c \neq \delta}{x \,||\, y \xrightarrow{c}_u x'}$$

$$\text{Par8}: \frac{x \xrightarrow{a}_u \surd \quad y \xrightarrow{b}_u \surd \quad a \mid b = c \quad c \neq \delta}{x \,||\, y \xrightarrow{c}_u \surd}$$

$$\begin{aligned}
&\mathcal{U}(\surd, 0)\\
&\mathcal{U}(a(u), v) \ \textit{if} \ v \leq u\\
&\mathcal{U}(\delta(u), v) \ \textit{if} \ v \leq u\\
&\mathcal{U}(x \cdot y, v) \Leftrightarrow \mathcal{U}(x, v)\\
&\mathcal{U}(x + y, v) \Leftrightarrow \mathcal{U}(x, v) \vee \mathcal{U}(y, v)\\
&\mathcal{U}(x \,||\, y, v) \Leftrightarrow \mathcal{U}(x, v) \wedge \mathcal{U}(y, v)
\end{aligned}$$

In order to obtain a clean BNF grammar, process terms with decreasing time stamps, like $a(2)\cdot b(1)$, are allowed. Note that this process term is timed branching bisimilar to $a(2)\cdot\delta(2)$.

6.3. Congruence Proof for Sequential Composition

First, we prove that rooted timed branching bisimilarity is a congruence for the sequential composition operator (see Theorem 6.1).

We give an example to show that if $p_0 \leftrightarrow_{tb}^{u} q_0$ and $p_1 \leftrightarrow_{tb}^{u} q_1$, then not necessarily $p_0 \cdot p_1 \leftrightarrow_{tb}^{u} q_0 \cdot q_1$.

Example 6.1. Let $p_0 = q_0 = a(1)$, $p_1 = \tau(0)\cdot b(1)$, $q_1 = b(1)$, and $u = 0$. Clearly, $a(1) \leftrightarrow_{tb}^{0} a(1)$. Also, $\tau(0)\cdot b(1) \leftrightarrow_{tb}^{0} b(1)$. However, clearly, $a(1)\cdot\tau(0)\cdot b(1) \leftrightarrow_{tb}^{0} a(1)\cdot b(1)$ does not hold.

From Example 6.1, it follows that the standard approach to prove that untimed rooted branching bisimilarity is a congruence, i.e. take the smallest congruence closure and prove that this yields a branching bisimulation (see [5]), fails for timed rooted branching bisimilarity when considering sequential composition. This motivates the usage of $p_1 \leftrightarrow_{rtb} q_1$ in Definition 6.2.

Definition 6.2. (Relation C_u)
Let $C_u \subseteq \mathcal{S} \times \mathcal{S}$ for $u \in \mathit{Time}$ denote the smallest relation such that:

1. $\leftrightarrows_{tb}^{u} \subseteq C_u$;
2. if $p_0 \; C_u \; q_0$ and $p_1 \leftrightarrows_{rtb} q_1$, then $p_0{\cdot}p_1 \; C_u \; q_0{\cdot}q_1$;
3. if $p_0 \; C_u \surd$ and $p_1 \leftrightarrows_{rtb} q_1$, then $p_0{\cdot}p_1 \; C_u \; q_1$;
4. if $\surd \; C_u \; q_0$ and $p_1 \leftrightarrows_{rtb} q_1$, then $p_1 \; C_u \; q_0{\cdot}q_1$.

The proof of the following key lemma is presented in the appendix, in Section A.1.

Lemma 6.1. The relations C_u constitute a timed branching bisimulation.

Theorem 6.1. If $p_0 \leftrightarrows_{rtb} q_0$ and $p_1 \leftrightarrows_{rtb} q_1$, then $p_0{\cdot}p_1 \leftrightarrows_{rtb} q_0{\cdot}q_1$.

Proof:
By Definition 6.1, we distinguish four cases:

1. Let $p_0{\cdot}p_1 \xrightarrow{\ell}_u s$. By the transition rules, we can distinguish two cases:
   (a) $p_0 \xrightarrow{\ell}_u p_0'$ and $s = p_0'{\cdot}p_1$. Since $p_0 \leftrightarrows_{rtb} q_0$, $q_0 \xrightarrow{\ell}_u t$ with $p_0' \leftrightarrows_{tb}^{u} t$. By the transition rules, we can distinguish two cases:
      i. Either $t \neq \surd$ and $q_0{\cdot}q_1 \xrightarrow{\ell}_u t{\cdot}q_1$. By Lemma 6.1, $p_0'{\cdot}p_1 \leftrightarrows_{tb}^{u} t{\cdot}q_1$.
      ii. Or $t = \surd$ and $q_0{\cdot}q_1 \xrightarrow{\ell}_u q_1$. By Lemma 6.1, $p_0'{\cdot}p_1 \leftrightarrows_{tb}^{u} q_1$.
   (b) $p_0 \xrightarrow{\ell}_u \surd$ and $s = p_1$. Since $p_0 \leftrightarrows_{rtb} q_0$, $q_0 \xrightarrow{\ell}_u t$ with $\surd \leftrightarrows_{tb}^{u} t$. By the transition rules, we can distinguish two cases:
      i. Either $t \neq \surd$ and $q_0{\cdot}q_1 \xrightarrow{\ell}_u t{\cdot}q_1$. By Lemma 6.1, $p_1 \leftrightarrows_{tb}^{u} t{\cdot}q_1$.
      ii. Or $t = \surd$ and $q_0{\cdot}q_1 \xrightarrow{\ell}_u q_1$. Since $p_1 \leftrightarrows_{rtb} q_1$, $p_1 \leftrightarrows_{tb}^{u} q_1$.
2. Let $q_0{\cdot}q_1 \xrightarrow{\ell}_u t$. Similar to the previous case.
3. Let $\mathcal{U}(p_0{\cdot}p_1, u)$. Then $\mathcal{U}(p_0, u)$. Since $p_0 \leftrightarrows_{rtb} q_0$, $\mathcal{U}(q_0, u)$. This means that $\mathcal{U}(q_0{\cdot}q_1, u)$.
4. Let $\mathcal{U}(q_0{\cdot}q_1, u)$. Similar to the previous case.

□

6.4. Congruence Proof for Alternative Composition

Next, we prove that rooted timed branching bisimilarity is a congruence for the alternative composition operator.

Theorem 6.2. If $p_0 \leftrightarrows_{rtb} q_0$ and $p_1 \leftrightarrows_{rtb} q_1$, then $p_0 + p_1 \leftrightarrows_{rtb} q_0 + q_1$.

Proof:
By Definition 6.1, we distinguish four cases:

1. Let $p_0 + p_1 \xrightarrow{\ell}_u s$. By the transition rules, we can distinguish two cases:
   (a) $p_0 \xrightarrow{\ell}_u s$. Since $p_0 \leftrightarroweq_{rtb} q_0$, $q_0 \xrightarrow{\ell}_u t$ with $s \leftrightarroweq^u_{tb} t$. Then $q_0 + q_1 \xrightarrow{\ell}_u t$.
   (b) $p_1 \xrightarrow{\ell}_u s$. Similar to the previous case.
2. Let $q_0 + q_1 \xrightarrow{\ell}_u t$. Similar to the previous case.
3. Let $\mathcal{U}(p_0 + p_1, u)$. Since $\mathcal{U}(p_0 + p_1, u)$, either $\mathcal{U}(p_0, u)$ or $\mathcal{U}(p_1, u)$. Since $p_0 \leftrightarroweq_{rtb} q_0$ and $p_1 \leftrightarroweq_{rtb} q_1$, either $\mathcal{U}(q_0, u)$, or $\mathcal{U}(q_1, u)$, respectively. Hence, $\mathcal{U}(q_0 + q_1, u)$.
4. Let $\mathcal{U}(q_0 + q_1, u)$. Similar to the previous case.

□

6.5. Congruence Proof for Parallel Composition

Finally, we indicate how to prove that rooted timed branching bisimilarity is a congruence for the parallel composition operator. This proof largely follows the one for sequential composition.

Definition 6.3. (Relation D_u)
Let $D_u \subseteq \mathcal{S} \times \mathcal{S}$ for $u \in \mathit{Time}$ denote the smallest relation such that:

1. $\leftrightarroweq^u_{tb} \subseteq D_u$;
2. if $p_0\ D_u\ q_0$ and $p_1\ D_u\ q_1$, then $p_0 \parallel p_1\ D_u\ q_0 \parallel q_1$;
3. if $p_0\ D_u\ \surd$ and $p_1\ D_u\ q_1$, then $p_0 \parallel p_1\ D_u\ q_1$;
4. if $p_0\ D_u\ q_0$ and $p_1\ D_u\ \surd$, then $p_0 \parallel p_1\ D_u\ q_0$;
5. if $\surd\ D_u\ q_0$ and $p_1\ D_u\ q_1$, then $p_1\ D_u\ q_0 \parallel q_1$;
6. if $p_0\ D_u\ q_0$ and $\surd\ D_u\ q_1$, then $p_0\ D_u\ q_0 \parallel q_1$;
7. if $p_0\ D_u\ \surd$ and $p_1\ D_u\ \surd$, then $p_0 \parallel p_1\ D_u\ \surd$;
8. if $\surd\ D_u\ q_0$ and $\surd\ D_u\ q_1$, then $\surd\ D_u\ q_0 \parallel q_1$.

The proofs of the following two lemmas are presented in the appendix, in Sections A.2 and A.3. For the sake of presentation, in the proofs of Lemma 6.3 and Theorem 6.3, all cases that involve successful termination have been discarded.

Lemma 6.2. If $p\ D_u\ q$ and $\mathcal{U}(p, u)$, then $\mathcal{U}(q, u)$.

Lemma 6.3. The relations D_u constitute a timed branching bisimulation.

Theorem 6.3. If $p_0 \leftrightarroweq_{rtb} q_0$ and $p_1 \leftrightarroweq_{rtb} q_1$, then $p_0 \parallel p_1 \leftrightarroweq_{rtb} q_0 \parallel q_1$.

Proof:
By Defi nition 6.1, we distinguish four cases:

1. Let $p_0 \mid\mid p_1 \xrightarrow{\ell}_u s$. By the transition rules, we can distinguish three cases (eight if we consider successful termination):

   (a) $p_0 \xrightarrow{\ell}_u p_0'$ and $s = p_0' \mid\mid p_1$. Since $p_0 \leftrightarrows_{rtb} q_0$, $q_0 \xrightarrow{\ell}_u t$ with $p_0' \leftrightarrows_{tb}^u t$. Since we do not consider successful termination ($t \neq \surd$), by the transition rules, $q_0 \mid\mid q_1 \xrightarrow{\ell}_u t \mid\mid q_1$. By Lemma 6.3, $p_0' \mid\mid p_1 \leftrightarrows_{tb}^u t \mid\mid q_1$.

   (b) $p_1 \xrightarrow{\ell}_u p_1'$ and $s = p_0 \mid\mid p_1'$. Similar to the previous case.

   (c) $p_0 \mid\mid p_1 \xrightarrow{\ell}_u p_0' \mid\mid p_1'$ with $\ell_0, \ell_1 \in Act$ such that $p_0 \xrightarrow{\ell_0}_u p_0'$, $p_1 \xrightarrow{\ell_1}_u p_1'$, and $\ell_0 \mid \ell_1 = \ell$. Since $p_0 \leftrightarrows_{rtb} q_0$, $q_0 \xrightarrow{\ell_0}_u t_0$ with $p_0' \leftrightarrows_{rtb}^u t_0$. Since $p_1 \leftrightarrows_{rtb} q_1$, $q_1 \xrightarrow{\ell_0}_u t_1$ with $p_1' \leftrightarrows_{rtb}^u t_1$. Since we do not consider successful termination ($t_0 \neq \surd$ and $t_1 \neq \surd$), by the transition rules, $q_0 \mid\mid q_1 \xrightarrow{\ell}_u t_0 \mid\mid t_1$. By Lemma 6.3, $p_0' \mid\mid p_1' \leftrightarrows_{tb}^u t_0 \mid\mid t_1$.

2. Let $q_0 \mid\mid q_1 \xrightarrow{\ell}_u t$. Similar to the previous case.

3. Let $\mathcal{U}(p_0 \mid\mid p_1, u)$. Then $\mathcal{U}(p_0, u)$ and $\mathcal{U}(p_1, u)$. Since $p_0 \leftrightarrows_{rtb} q_0$ and $p_1 \leftrightarrows_{rtb} q_1$, $\mathcal{U}(q_0, u)$ and $\mathcal{U}(q_1, u)$. This means that $\mathcal{U}(q_0 \mid\mid q_1, u)$.

4. Let $\mathcal{U}(q_0 \mid\mid q_1, u)$. Similar to the previous case.

□

7. Conclusions

Equivalence and congruence properties for timed semantics are often claimed, but hardly ever proved. In this paper, we showed that this is a dangerous practice: two closely related definitions for timed branching bisimilarity are shown to violate transitivity, in case of a dense time domain. We resolved this problem by strengthening the semantic definition; the timed branching bisimulation relation must be established explicitly when time progresses. We showed that in case of a discrete time domain, the earlier notion of timed branching bisimilarity by Van der Zwaag and our strengthened notion coincide. Finally, we went on to prove that our notion constitutes a congruence, for a simple timed process algebra with timed actions and alternative/sequential/parallel composition.

Does our strengthened notion have practical relevance? Probably not. The strengthened condition, that the timed branching bisimulation relation must be established explicitly when time progresses, makes it very hard to check our version of timed branching bisimilarity. But our results do offer useful insights into timed semantics: why transitivity may fail, and how equivalence and congruence can be proved, for weak versions of timed bisimulation.

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A. Proofs of Three Lemmas

This appendix contains the proofs of three lemmas for the congruence result.

A.1. Proof of Lemma 6.1

The proof consists of three parts (plus three symmetric parts).

A *If* $s\ C_u\ t$ *and* $s \xrightarrow{\ell}_u s'$*, then we must prove, that*

- i *either* $\ell = \tau$ *and* $s'\ C_u\ t$*,*
- ii *or* $t \Rightarrow_u \hat{t} \xrightarrow{\ell}_u t'$ *with* $s\ C_u\ \hat{t}$ *and* $s'\ C_u\ t'$*.*

We apply induction on the structure of s and t. Since $s\ C_u\ t$, by Definition 6.2, we can distinguish four cases.

Firstly, $s \underline{\leftrightarrow}_{tb}^{u} t$. The proof obligation follows directly from the definition of timed branching bisimilarity.

Secondly, $s = p_0{\cdot}p_1$ and $t = q_0{\cdot}q_1$, with $p_0\ C_u\ q_0$ and $p_1 \underline{\leftrightarrow}_{rtb} q_1$. Since $p_0{\cdot}p_1 \xrightarrow{\ell}_u s'$, by the transition rules, we can distinguish two cases:

1. Let $p_0 \xrightarrow{\ell}_u p_0'$ and $s' = p_0'{\cdot}p_1$. Since $p_0\ C_u\ q_0$, by Definition 6.2, we can distinguish four cases:
   - (a) $p_0 \underline{\leftrightarrow}_{tb}^{u} q_0$. Since $p_0 \xrightarrow{\ell}_u p_0'$, by Definition 4.1, we can again distinguish two cases:
     - i $\ell = \tau$ and $p_0' \underline{\leftrightarrow}_{tb}^{u} q_0$. Then $p_0'{\cdot}p_1\ C_u\ q_0{\cdot}q_1$.
     - ii $q_0 \Rightarrow_u \hat{q} \xrightarrow{\ell}_u t_0$, with $p_0 \underline{\leftrightarrow}_{tb}^{u} \hat{q}$ and $p_0' \underline{\leftrightarrow}_{tb}^{u} t_0$. Then,
       - ∗ either $t_0 \neq \surd$ and $q_0{\cdot}q_1 \Rightarrow_u \hat{q}{\cdot}q_1 \xrightarrow{\ell}_u t_0{\cdot}q_1$, $p_0{\cdot}p_1\ C_u\ \hat{q}{\cdot}q_1$ and $p_0'{\cdot}p_1\ C_u\ t_0{\cdot}q_1$;
       - ∗ or $t_0 = \surd$ and $q_0{\cdot}q_1 \Rightarrow_u \hat{q}{\cdot}q_1 \xrightarrow{\ell}_u q_1$, $p_0{\cdot}p_1\ C_u\ \hat{q}{\cdot}q_1$ and $p_0'{\cdot}p_1\ C_u\ q_1$.
   - (b) $p_0 = p_{00}{\cdot}p_{01}$ and $q_0 = q_{00}{\cdot}q_{01}$ with $p_{00}\ C_u\ q_{00}$ and $p_{01} \underline{\leftrightarrow}_{rtb} q_{01}$. Since $p_{00}{\cdot}p_{01} \xrightarrow{\ell}_u p_0'$, by the transition rules, either $p_{00} \xrightarrow{\ell}_u p_{00}'$ with $p_0' = p_{00}'{\cdot}p_{01}$, or $p_{00} \xrightarrow{\ell}_u \surd$ and $p_0' = p_{01}$.
     In the first case, since $p_{00}\ C_u\ q_{00}$ and $p_{00} \xrightarrow{\ell}_u p_{00}'$, by induction,
     - i either $\ell = \tau$ and $p_{00}'\ C_u\ q_{00}$. Then $p_{00}'{\cdot}p_{01}{\cdot}p_1\ C_u\ q_{00}{\cdot}q_{01}{\cdot}q_1$.
     - ii or $q_{00} \Rightarrow_u \hat{q}_{00} \xrightarrow{\ell}_u t_{00}$, with $p_{00}\ C_u\ \hat{q}_{00}$ and $p_{00}'\ C_u\ t_{00}$. Then,
       - ∗ either $t_{00} \neq \surd$ and $q_{00}{\cdot}q_{01} \Rightarrow_u \hat{q}_{00}{\cdot}q_{01} \xrightarrow{\ell}_u t_{00}{\cdot}q_{01}$, so $q_{00}{\cdot}q_{01}{\cdot}q_1 \Rightarrow_u \hat{q}_{00}{\cdot}q_{01}{\cdot}q_1 \xrightarrow{\ell}_u t_{00}{\cdot}q_{01}{\cdot}q_1$. Furthermore $p_{00}{\cdot}p_{01}{\cdot}p_1\ C_u\ \hat{q}_{00}{\cdot}q_{01}{\cdot}q_1$ and $p_{00}'{\cdot}p_{01}{\cdot}p_1\ C_u\ t_{00}{\cdot}q_{01}{\cdot}q_1$.
       - ∗ or $t_{00} = \surd$ and $q_{00}{\cdot}q_{01} \Rightarrow_u \hat{q}_{00}{\cdot}q_{01} \xrightarrow{\ell}_u q_{01}$, so $q_{00}{\cdot}q_{01}{\cdot}q_1 \Rightarrow_u \hat{q}_{00}{\cdot}q_{01}{\cdot}q_1 \xrightarrow{\ell}_u q_{01}{\cdot}q_1$. Furthermore $p_{00}{\cdot}p_{01}{\cdot}p_1\ C_u\ \hat{q}_{00}{\cdot}q_{01}{\cdot}q_1$ and $p_{00}'{\cdot}p_{01}{\cdot}p_1\ C_u\ q_{01}{\cdot}q_1$.

     In the second case, since $p_{00}\ C_u\ q_{00}$ and $p_{00} \xrightarrow{\ell}_u \surd$, by induction,
     - i either $\ell = \tau$ and $\surd\ C_u\ q_{00}$. Then $p_{01}{\cdot}p_1\ C_u\ q_{00}{\cdot}q_{01}{\cdot}q_1$.
     - ii or $q_{00} \Rightarrow_u \hat{q}_{00} \xrightarrow{\ell}_u t_{00}$ with $p_{00}\ C_u\ \hat{q}_{00}$ and $\surd\ C_u\ t_{00}$. Then,

- $*$ either $t_{00} \neq \surd$ and $q_{00}{\cdot}q_{01}{\cdot}q_1 \Longrightarrow_u \hat{q}_{00}{\cdot}q_{01}{\cdot}q_1 \xrightarrow{\ell}_u t_{00}{\cdot}q_{01}{\cdot}q_1$. Furthermore $p_{00}{\cdot}p_{01}{\cdot}p_1 \; C_u \; \hat{q}_{00}{\cdot}q_{01}{\cdot}q_1$ and $p_{01}{\cdot}p_1 \; C_u \; t_{00}{\cdot}q_{01}{\cdot}q_1$.
- $*$ or $t_{00} = \surd$ and $q_{00}{\cdot}q_{01}{\cdot}q_1 \Longrightarrow_u \hat{q}_{00}{\cdot}q_{01}{\cdot}q_1 \xrightarrow{\ell}_u q_{01}{\cdot}q_1$. Furthermore $p_{00}{\cdot}p_{01}{\cdot}p_1 \; C_u \; \hat{q}_{00}{\cdot}q_{01}{\cdot}q_1$ and $p_{01}{\cdot}p_1 \; C_u \; q_{01}{\cdot}q_1$.

(c) $p_0 = p_{00}{\cdot}p_{01}$ with $p_{00} \; C_u \surd$ and $p_{01} \leftrightarrow_{rtb} q_0$. Since $p_{00}{\cdot}p_{01} \xrightarrow{\ell}_u p_0'$, by the transition rules, either $p_{00} \xrightarrow{\ell}_u p_{00}'$ with $p_0' = p_{00}'{\cdot}p_{01}$, or $p_{00} \xrightarrow{\ell}_u \surd$ and $p_0' = p_{01}$. In the first case, since $p_{00} \; C_u \surd$, by induction, $\ell = \tau$ and $p_{00}' \; C_u \surd$. Then $p_{00}'{\cdot}p_{01}{\cdot}p_1 \; C_u \; q_0{\cdot}q_1$. In the second case, since $p_{00} \; C_u \surd$, by induction, $\ell = \tau$. Moreover, $p_{01}{\cdot}p_1 \; C_u \; q_0{\cdot}q_1$.

(d) $q_0 = q_{00}{\cdot}q_{01}$ with $\surd \; C_u \; q_{00}$ and $p_0 \leftrightarrow_{rtb} q_{01}$. Since $p_0 \leftrightarrow_{rtb} q_{01}$ and $p_0 \xrightarrow{\ell}_u p_0'$, by induction, $q_{01} \xrightarrow{\ell}_u q_{01}'$ with $p_0' \leftrightarrow_{tb}^u q_{01}'$. Then $p_0'{\cdot}p_1 \; C_u \; q_{00}{\cdot}q_{01}'{\cdot}q_1$.

2. Let $p_0 \xrightarrow{\ell}_u \surd$ and $s' = p_1$. Since $p_0 \; C_u \; q_0$, by induction,

   i either $\ell = \tau$ and $\surd \; C_u \; q_0$. Then $p_1 \; C_u \; q_0{\cdot}q_1$.

   ii or $q_0 \Longrightarrow_u \hat{q}_0 \xrightarrow{\ell}_u t_0$ with $p_0 \; C_u \; \hat{q}_0$ and $\surd \; C_u \; t_0$. Then,

   - $*$ either $t_0 \neq \surd$ and $q_0{\cdot}q_1 \Longrightarrow_u \hat{q}_0{\cdot}q_1 \xrightarrow{\ell}_u t_0{\cdot}q_1$. Furthermore $p_0{\cdot}p_1 \; C_u \; \hat{q}_0{\cdot}q_1$ and $p_1 \; C_u \; t_0{\cdot}q_1$.
   - $*$ or $t_0 = \surd$ and $q_0{\cdot}q_1 \Longrightarrow_u \hat{q}_0{\cdot}q_1 \xrightarrow{\ell}_u q_1$. Furthermore $p_0{\cdot}p_1 \; C_u \; \hat{q}_0{\cdot}q_1$ and $p_1 \; C_u \; q_1$.

Thirdly, $s = p_0{\cdot}p_1$, with $p_0 \; C_u \surd$ and $p_1 \leftrightarrow_{rtb} t$. Since $p_0{\cdot}p_1 \xrightarrow{\ell}_u s'$, by the transition rules, we can distinguish two cases:

1. $p_0 \xrightarrow{\ell}_u p_0'$ and $s' = p_0'{\cdot}p_1$. Since $p_0 \; C_u \surd$, by induction, $\ell = \tau$ and $p_0' \; C_u \surd$. Then $p_0'{\cdot}p_1 \; C_u \; t$.
2. $p_0 \xrightarrow{\ell}_u \surd$ and $s' = p_1$. Since $p_0 \; C_u \surd$, by induction, $\ell = \tau$. And $p_1 \leftrightarrow_{rtb} t$ clearly implies $p_1 \; C_u \; t$.

Fourthly, $t = q_0{\cdot}q_1$, with $\surd \; C_u \; q_0$ and $s \leftrightarrow_{rtb} q_1$. Since $\surd\downarrow$, by induction, $q_0 \Longrightarrow_u t_0 \downarrow$. Clearly, $t_0 = \surd$. Since $s \leftrightarrow_{rtb} q_1$, $s \xrightarrow{\ell}_u s'$ implies $q_1 \xrightarrow{\ell}_u t'$ with $s' \leftrightarrow_{tb}^u t'$. So $q_0{\cdot}q_1 \Longrightarrow_u q_1 \xrightarrow{\ell}_u t'$.

B *If $s \; C_u \; t$ and $s\downarrow$, then we must prove, that $t \Longrightarrow_u t'\downarrow$, with $s \; C_u \; t'$.*

We apply induction on the structure of s and t. Since $s \; C_u \; t$, by Definition 6.2, we can distinguish four cases:

Firstly, $s \leftrightarrow_{tb}^u t$. The proof obligation follows directly from the definition of timed branching bisimilarity.

Secondly, $s = p_0{\cdot}p_1$ and $t = q_0{\cdot}q_1$, with $p_0 \; C_u \; q_0$ and $p_1 \leftrightarrow_{rtb} q_1$. This case is vacuous, since $s = p_0{\cdot}p_1$ contradicts $s\downarrow$.

Thirdly, $s = p_0{\cdot}p_1$, with $p_0 \; C_u \surd$ and $p_1 \leftrightarrow_{rtb} t$. This case is vacuous, since $s = p_0{\cdot}p_1$ contradicts $s\downarrow$.

Fourthly, $t = q_0 \cdot q_1$, with $\surd\; C_u\; q_0$ and $s \leftrightarroweq_{rtb} q_1$. This case is vacuous, since $s \downarrow$ and $s \leftrightarroweq_{rtb} q_1$ implies $q_1 \downarrow$, which is not possible.

C *If $s\; C_u\; t$ and $u \leq v$ and $\mathcal{U}(s, v)$, then we must prove, that for some $n \geq 0$ there are $t_0, \ldots, t_n \in \mathcal{S}$ with $t = t_0$ and $\mathcal{U}(t_n, v)$, and $u_0, \ldots, u_n \in \mathit{Time}$ with $u = u_0$ and $v = u_n$, such that for $i < n$, $t_i \Longrightarrow_{u_i} t_{i+1}$ and $s\; C_w\; t_{i+1}$ for $u_i \leq w \leq u_{i+1}$.*

We apply induction on the structure of s and t. Since $s\; C_u\; t$, by Definition 6.2, we can distinguish four cases:

Firstly, $s \leftrightarroweq^u_{tb} t$. The proof obligation follows directly from the definition of timed branching bisimilarity.

Secondly, $s = p_0 \cdot p_1$ and $t = q_0 \cdot q_1$, with $p_0\; C_u\; q_0$ and $p_1 \leftrightarroweq_{rtb} q_1$. Since $\mathcal{U}(p_0 \cdot p_1,\, v)$, also $\mathcal{U}(p_0, v)$. Since $p_0\; C_u\; q_0$ and $u \leq v$, by induction, for some $n \geq 0$ there are $\hat{q}_0, \ldots, \hat{q}_n \in \mathcal{S}$ with $q_0 = \hat{q}_0$ and $\mathcal{U}(\hat{q}_n, v)$, and $u_0, \ldots, u_n \in \mathit{Time}$ with $u = u_0$ and $v = u_n$, such that for $i < n$, $\hat{q}_i \Longrightarrow_{u_i} \hat{q}_{i+1}$ and $p_0\; C_w\; \hat{q}_{i+1}$ for $u_i \leq w \leq u_{i+1}$. Clearly, $t = \hat{q}_0 \cdot q_1$ and $\mathcal{U}(\hat{q}_n \cdot q_1, v)$, and for $i < n$, $\hat{q}_i \cdot q_1 \Longrightarrow_{u_i} \hat{q}_{i+1} \cdot q_1$ and $p_0 \cdot p_1\; C_w\; \hat{q}_{i+1} \cdot q_1$ for $u_i \leq w \leq u_{i+1}$.

Thirdly, $s = p_0 \cdot p_1$ with $p_0\; C_u\; \surd$ and $p_1 \leftrightarroweq_{rtb} t$. Since $\mathcal{U}(s, v)$, also $\mathcal{U}(p_0, v)$. Since $v > u$, by case C, this contradicts $p_0\; C_u\; \surd$, so this case is vacuous.

Fourthly, $t = q_0 \cdot q_1$ with $\surd\; C_u\; q_0$ and $s \leftrightarroweq_{rtb} q_1$. Since $\surd\; C_u\; q_0$, by case C, $q_0 \Longrightarrow_u \surd$ and $\neg\mathcal{U}(q_0, v)$ for $v \geq u$. So $q_0 \cdot q_1 \Longrightarrow_u q_1$. Since $s \leftrightarroweq_{rtb} q_1$ and $\mathcal{U}(s, v)$, also $\mathcal{U}(q_1, v)$. Clearly, the proof obligation holds with $n = 1$.

A.2. Proof of Lemma 6.2

By Definition 6.3, we can distinguish six cases (the last two of Definition 6.3 are not applicable):

1. $p \leftrightarroweq^u_{tb} q$. Then it follows immediately from Definition 4.1, case 6, that $\mathcal{U}(q, u)$.[5]

2. $p = p_0 \,||\, p_1$ and $q = q_0 \,||\, q_1$, with $p_0\; D_u\; q_0$ and $p_1\; D_u\; q_1$. Since $\mathcal{U}(p_0 \,||\, p_1, u)$, we have $\mathcal{U}(p_0, u)$ and $\mathcal{U}(p_1, u)$. Therefore, by induction, $\mathcal{U}(q_0, u)$ and $\mathcal{U}(q_1, u)$. From this, it follows that $\mathcal{U}(q_0 \,||\, q_1, u)$.

3. $p = p_0 \,||\, p_1$, with $p_0\; D_u\; \surd$ and $p_1\; D_u\; q$. Since $\mathcal{U}(p_0 \,||\, p_1, u)$, we have $\mathcal{U}(p_1, u)$. Therefore, by induction, $\mathcal{U}(q, u)$.

4. $p = p_0 \,||\, p_1$, with $p_0\; D_u\; q$ and $p_1\; D_u\; \surd$. Similar to the previous case.

5. $q = q_0 \,||\, q_1$, with $\surd\; D_u\; q_0$ and $p\; D_u\; q_1$. By Definition 6.3, $\surd \leftrightarroweq^u_{tb} q_0$. Since $q_0 \neq \surd$, it follows from Definition 4.1, case 3, that $\mathcal{U}(q_0, u)$. Since $\mathcal{U}(p, u)$, by induction, $\mathcal{U}(q_1, u)$. From $\mathcal{U}(q_0, u)$ and $\mathcal{U}(q_1, u)$, it follows that $\mathcal{U}(q_0 \,||\, q_1, u)$.

6. $q = q_0 \,||\, q_1$, with $p\; D_u\; q_0$ and $\surd\; D_u\; q_1$. Similar to the previous case.

[5] Note that this holds due to '$u \leq v$' and '$n \geq 0$' in case 6, because in the original definition of Van der Zwaag, we would have $\delta(1) \leftrightarroweq^{Z,2}_{tb} \delta(2)$; see footnote 3.

A.3. Proof of Lemma 6.3

In this proof of Lemma 6.3, all cases that involve successful termination have been discarded. We point out that the full proof contains at least 528 different cases.

The proof consists of two parts (plus two symmetric parts); the proof consists of three parts (plus three symmetric parts) if we would take into account successful termination.

A *If $s\ D_u\ t$ and $s \xrightarrow{\ell}_u s'$, then we must prove, that*

i *either $\ell = \tau$ and $s'\ D_u\ t$,*

ii *or $t \Longrightarrow_u \hat{t} \xrightarrow{\ell}_u t'$ with $s\ D_u\ \hat{t}$ and $s'\ D_u\ t'$.*

We apply induction on the structure of s and t. Since $s\ D_u\ t$, by Definition 6.3, we can distinguish two cases (eight if we consider successful termination).

Firstly, $s \leftrightarrows_{tb}^u t$. The proof obligation follows directly from the definition of timed branching bisimilarity.

Secondly, $s = p_0 \mid\mid p_1$ and $t = q_0 \mid\mid q_1$, with $p_0\ D_u\ q_0$ and $p_1\ D_u\ q_1$. Since $p_0 \mid\mid p_1 \xrightarrow{\ell}_u s'$, by the transition rules, we can distinguish three cases (eight if we consider successful termination):

1. Let $p_0 \xrightarrow{\ell}_u p_0'$, $\mathcal{U}(p_1, u)$ and $s' = p_0' \mid\mid p_1$. By Lemma 6.2, since $\mathcal{U}(p_1, u)$, also $\mathcal{U}(q_1, u)$. Since $p_0\ D_u\ q_0$, by Definition 6.3, we can distinguish two cases (eight if we consider successful termination):

   (a) $p_0 \leftrightarrows_{tb}^u q_0$. Since $p_0 \xrightarrow{\ell}_u p_0'$, by Definition 4.1, we can again distinguish two cases:

   i $\ell = \tau$ and $p_0' \leftrightarrows_{tb}^u q_0$. Then $p_0' \mid\mid p_1\ D_u\ q_0 \mid\mid q_1$.

   ii $q_0 \Longrightarrow_u \hat{q} \xrightarrow{\ell}_u t_0$, with $p_0 \leftrightarrows_{tb}^u \hat{q}$ and $p_0' \leftrightarrows_{tb}^u t_0$. Then, since $\mathcal{U}(q_1, u)$ (and we do not consider successful termination, hence $t_0 \neq \surd$), it follows that $q_0 \mid\mid q_1 \Longrightarrow_u \hat{q} \mid\mid q_1 \xrightarrow{\ell}_u t_0 \mid\mid q_1$, $p_0 \mid\mid p_1\ D_u\ \hat{q} \mid\mid q_1$ and $p_0' \mid\mid p_1\ D_u\ t_0 \mid\mid q_1$.

   (b) $p_0 = p_{00} \mid\mid p_{01}$ and $q_0 = q_{00} \mid\mid q_{01}$ with $p_{00}\ D_u\ q_{00}$ and $p_{01}\ D_u\ q_{01}$. Since $p_{00} \mid\mid p_{01} \xrightarrow{\ell}_u p_0'$, by the transition rules (not considering successful termination), either $p_{00} \xrightarrow{\ell}_u p_{00}'$ and $\mathcal{U}(p_{01}, u)$ with $p_0' = p_{00}' \mid\mid p_{01}$, or $p_{01} \xrightarrow{\ell}_u p_{01}'$ and $\mathcal{U}(p_{00}, u)$ with $p_0' = p_{00} \mid\mid p_{01}'$, or $p_{00} \mid\mid p_{01} \xrightarrow{\ell}_u p_{00}' \mid\mid p_{01}'$ with $p_0' = p_{00}' \mid\mid p_{01}'$ if there exist $\ell_0, \ell_1 \in Act$ such that $p_{00} \xrightarrow{\ell_0}_u p_{00}'$, $p_{01} \xrightarrow{\ell_1}_u p_{01}'$, and $\ell_0 \mid \ell_1 = \ell$.
   In the first case, by Lemma 6.2, since $\mathcal{U}(p_{01}, u)$, also $\mathcal{U}(q_{01}, u)$. Since $p_{00}\ D_u\ q_{00}$ and $p_{00} \xrightarrow{\ell}_u p_{00}'$, by induction,

   i either $\ell = \tau$ and $p_{00}'\ D_u\ q_{00}$. Then $p_{00}' \mid\mid p_{01} \mid\mid p_1\ D_u\ q_{00} \mid\mid q_{01} \mid\mid q_1$.

   ii or $q_{00} \Longrightarrow_u \hat{q}_{00} \xrightarrow{\ell}_u t_{00}$, with $p_{00}\ D_u\ \hat{q}_{00}$ and $p_{00}'\ D_u\ t_{00}$. Then (we do not consider successful termination here, hence $t_{00} \neq \surd$), $q_{00} \mid\mid q_{01} \Longrightarrow_u \hat{q}_{00} \mid\mid q_{01} \xrightarrow{\ell}_u t_{00} \mid\mid q_{01}$, so, since $\mathcal{U}(q_{01} \mid\mid q_1, u)$, $q_{00} \mid\mid q_{01} \mid\mid q_1 \Longrightarrow_u \hat{q}_{00} \mid\mid q_{01} \mid\mid q_1 \xrightarrow{\ell}_u t_{00} \mid\mid q_{01} \mid\mid q_1$. Furthermore $p_{00} \mid\mid p_{01} \mid\mid p_1\ D_u\ \hat{q}_{00} \mid\mid q_{01} \mid\mid q_1$ and $p_{00}' \mid\mid p_{01} \mid\mid p_1\ D_u\ t_{00} \mid\mid q_{01} \mid\mid q_1$.

The second case is similar to the first case.
In the third case, since $p_{00}\ D_u\ q_{00}$, $p_{00} \xrightarrow{\ell_0}_u p'_{00}$, and $\ell_0 \neq \tau$ (since $\ell_0 \mid \ell_1 = \ell$), by induction, $q_{00} \Longrightarrow_u \hat{q}_{00} \xrightarrow{\ell_0}_u t_{00}$, with $p_{00}\ D_u\ \hat{q}_{00}$ and $p'_{00}\ D_u\ t_{00}$. Similarly, since $p_{01}\ D_u\ q_{01}$, $p_{01} \xrightarrow{\ell_1}_u p'_{01}$, and $\ell_1 \neq \tau$ (since $\ell_0 \mid \ell_1 = \ell$), by induction, $q_{01} \Longrightarrow_u \hat{q}_{01} \xrightarrow{\ell_0}_u t_{01}$. Then (we do not consider successful termination here, hence $t_{00} \neq \surd$ and $t_{01} \neq \surd$), it follows that $q_{00} \mid\mid q_{01} \Longrightarrow_u \hat{q}_{00} \mid\mid \hat{q}_{01} \xrightarrow{\ell}_u t_{00} \mid\mid t_{01}$. Since $\mathcal{U}(q_1, u)$, $q_{00} \mid\mid q_{01} \mid\mid q_1 \Longrightarrow_u \hat{q}_{00} \mid\mid \hat{q}_{01} \mid\mid q_1 \xrightarrow{\ell}_u t_{00} \mid\mid t_{01} \mid\mid q_1$. Furthermore
$p_{00} \mid\mid p_{01} \mid\mid p_1\ D_u\ \hat{q}_{00} \mid\mid \hat{q}_{01} \mid\mid q_1$ and $p'_{00} \mid\mid p'_{01} \mid\mid p_1\ D_u\ t_{00} \mid\mid t_{01} \mid\mid q_1$.

2. Let $p_1 \xrightarrow{\ell}_u p'_1$, $\mathcal{U}(p_0, u)$ and $s' = p_0 \mid\mid p'_1$. This case is similar to the previous case.

3. Let $p_0 \mid\mid p_1 \xrightarrow{\ell}_u p'_0 \mid\mid p_1$ with $\ell_0, \ell_1 \in Act$ such that $p_0 \xrightarrow{\ell_0}_u p'_0$, $p_1 \xrightarrow{\ell_1}_u p'_1$, and $\ell_0 \mid \ell_1 = \ell$. Since $p_0\ D_u\ q_0$, $p_0 \xrightarrow{\ell_0}_u p'_0$, and $\ell_0 \neq \tau$ (since $\ell_0 \mid \ell_1 = \ell$), by induction, $q_0 \Longrightarrow_u \hat{q}_0 \xrightarrow{\ell_0}_u t_0$, with $p_0\ D_u\ \hat{q}_0$ and $p'_0\ D_u\ t_0$. Similarly, since $p_1\ D_u\ q_1$, $p_1 \xrightarrow{\ell_1}_u p'_1$, and $\ell_1 \neq \tau$ (since $\ell_0 \mid \ell_1 = \ell$), by induction, $q_1 \Longrightarrow_u \hat{q}_1 \xrightarrow{\ell_1}_u t_1$. Then (we do not consider successful termination here, hence $t_0 \neq \surd$ and $t_1 \neq \surd$), it follows that $q_0 \mid\mid q_1 \Longrightarrow_u \hat{q}_0 \mid\mid \hat{q}_1 \xrightarrow{\ell}_u t_0 \mid\mid t_1$. Furthermore $p_0 \mid\mid p_1\ D_u\ \hat{q}_0 \mid\mid \hat{q}_1$ and $p'_0 \mid\mid p'_1\ D_u\ t_0 \mid\mid t_1$.

B *If $s\ D_u\ t$ and $u \leq v$ and $\mathcal{U}(s, v)$, then we must prove, that for some $n \geq 0$ there are $t_0,\ldots,t_n \in \mathcal{S}$ with $t = t_0$ and $\mathcal{U}(t_n, v)$, and $u_0,\ldots,u_n \in$ Time with $u = u_0$ and $v = u_n$, such that for $i < n$, $t_i \Longrightarrow_{u_i} t_{i+1}$ and $s\ D_w\ t_{i+1}$ for $u_i \leq w \leq u_{i+1}$.*

We apply induction on the structure of s and t. Since $s\ D_u\ t$, by Definition 6.3, we can distinguish two cases (eight if we consider successful termination).

Firstly, $s \leftrightarrows^u_{tb} t$. The proof obligation follows directly from the definition of timed branching bisimilarity.

Secondly, $s = p_0 \mid\mid p_1$ and $t = q_0 \mid\mid q_1$, with $p_0\ D_u\ q_0$ and $p_1\ D_u\ q_1$. Since $\mathcal{U}(p_0 \mid\mid p_1, v)$, also $\mathcal{U}(p_0, v)$ and $\mathcal{U}(p_1, v)$. Since $p_0\ D_u\ q_0$ and $u \leq v$, by induction, for some $n \geq 0$ there are $\hat{q}_0, \ldots, \hat{q}_n \in \mathcal{S}$ with $q_0 = \hat{q}_0$ and $\mathcal{U}(\hat{q}_n, v)$, and $u_0, \ldots, u_n \in Time$ with $u = u_0$ and $v = u_n$, such that for $i < n$, $\hat{q}_i \Longrightarrow_{u_i} \hat{q}_{i+1}$ and $p_0\ D_w\ \hat{q}_{i+1}$ for $u_i \leq w \leq u_{i+1}$. Similarly, since $p_1\ D_u\ q_1$ and $u \leq v$, by induction, for some $m \geq 0$ there are $\hat{q}'_0, \ldots, \hat{q}'_m \in \mathcal{S}$ with $q_1 = \hat{q}'_0$ and $\mathcal{U}(\hat{q}'_m, v)$, and $u'_0, \ldots, u'_m \in Time$ with $u = u'_0$ and $v = u'_m$, such that for $i < m$, $\hat{q}'_i \Longrightarrow_{u'_i} \hat{q}'_{i+1}$ and $p_1\ D_w\ \hat{q}'_{i+1}$ for $u'_i \leq w \leq u'_{i+1}$. Clearly, these two sequences for $\hat{q}_0, \ldots, \hat{q}_n$ and for $\hat{q}'_0, \ldots, \hat{q}'_m$ can in a straightforward fashion be transformed into a sequence, such that for $k = n + m$ there are $\bar{q}_0, \ldots, \bar{q}_k \in \mathcal{S}$ and $\bar{q}'_0, \ldots, \bar{q}'_k \in \mathcal{S}$ with $q_0 = \bar{q}_0$, $\mathcal{U}(\bar{q}_k, v)$, $q_1 = \bar{q}'_0$, $\mathcal{U}(\bar{q}'_k, v)$, and $u_0, \ldots, u_k \in Time$ with $u = u_0$ and $v = u_k$, such that for $i < k$, $\bar{q}_i \mid\mid \bar{q}'_i \Longrightarrow_{u_i} \bar{q}_{i+1} \mid\mid \bar{q}'_{i+1}$ and $p_0 \mid\mid p_1\ D_w\ \bar{q}_{i+1} \mid\mid \bar{q}'_{i+1}$ for $u_i \leq w \leq u_{i+1}$.

B. Branching Tail Bisimulation

The notion of *branching tail bisimulation* from [2] is closely related to Van der Zwaag's definition of timed branching bisimulation. We show that in case of dense time, our counter-example (see Example 3.5) again shows that branching tail bisimilarity is not an equivalence relation.

In the absolute time setting of Baeten and Middelburg, states are of the form $<p, u>$ with p a process algebraic term and u a time stamp referring to the absolute time. They give an operational semantics to their process algebras such that if $<p, u> \stackrel{v}{\longmapsto} <p, u{+}v>$ (where $\stackrel{v}{\longmapsto}$ for $v > 0$ denotes a time step of v time units), then $<p, u> \stackrel{w}{\longmapsto} <p, u{+}w>$ for $0 < w < v$; in our example this saturation with time steps will be mimicked. The reflexive transitive closure of $\stackrel{\tau}{\longrightarrow}$ is denoted by $\twoheadrightarrow$. The relation $s \stackrel{u}{\mapsto\!\!\!\twoheadrightarrow} s'$ is defined by: either $s \twoheadrightarrow \hat{s} \stackrel{u}{\longmapsto} s'$, or $s \stackrel{v}{\mapsto\!\!\!\twoheadrightarrow} \hat{s} \stackrel{w}{\mapsto\!\!\!\twoheadrightarrow} s'$ with $v + w = u$.[6]

Branching tail bisimulation is defined as follows.[7]

Definition B.1. (Branching tail bisimulation [2])
Assume a TLTS in the style of Baeten and Middelburg. A symmetric binary relation $B \subseteq \mathcal{S} \times \mathcal{S}$ is a *branching tail bisimulation* if $s\ B\ t$ implies:

1. if $s \stackrel{\ell}{\longrightarrow} s'$, then

   i either $\ell = \tau$ and $t \twoheadrightarrow t'$ with $s\ B\ t'$ and $s'\ B\ t'$;

   ii or $t \twoheadrightarrow \hat{t} \stackrel{a}{\longrightarrow} t'$ with $s\ B\ \hat{t}$ and $s'\ B\ t'$;

2. if $s \stackrel{\ell}{\longrightarrow} <\surd, u>$, then $t \twoheadrightarrow t' \stackrel{\ell}{\longrightarrow} <\surd, u>$ with $s\ B\ t'$;

3. if $s \stackrel{u}{\longmapsto} s'$, then

   i either $t \twoheadrightarrow \hat{t} \stackrel{v}{\longmapsto} \hat{t}' \stackrel{w}{\mapsto\!\!\!\twoheadrightarrow} t'$ with $v + w = u$, $s\ B\ \hat{t}$ and $s'\ B\ t'$;

   ii or $t \twoheadrightarrow \hat{t} \stackrel{u}{\longmapsto} t'$ with $s\ B\ \hat{t}$ and $s'\ B\ t'$.

Two states s and t are *branching tail bisimilar*, written $s \leftrightarrow_{tb}^{BM} t$, if there is a branching tail bisimulation B with $s\ B\ t$.[8]

We proceed to transpose the TLTSs from Example 3.5 into the setting of Baeten and Middelburg. We now have the following transitions, for $i \geq 0$:

[6] Baeten and Middelburg also have a deadlock predicate $\uparrow$, which we do not take into account here, as it does not play a role in our counter-example.
[7] Baeten and Middelburg define this notion in the setting with relative time, and remark that the adaptation of this definition to absolute time is straightforward. Here we present this straightforward adaptation.
[8] The superscript *BM* refers to Baeten and Middelburg, to distinguish it from the notion of timed branching bisimulation as defined in Definition 4.1.

$$<p,0> \xrightarrow{\tau} <p_0,0>$$
$$<p_i,0> \xrightarrow{\tau} <p_{i+1},0>$$
$$<p_{i+1},0> \xrightarrow{\tau} <p_i,0>$$
$$<p_i,u> \overset{v-u}{\longmapsto} <p_i,v>,\ 0 \leq u < v \leq \tfrac{1}{i+2}$$
$$<p_i,\tfrac{1}{i+2}> \xrightarrow{\tau} <p'_i,\tfrac{1}{i+2}>$$
$$<p'_i,u> \overset{v-u}{\longmapsto} <p'_i,v>,\ \tfrac{1}{i+2} \leq u < v \leq 1$$
$$<p'_i,\tfrac{1}{n}> \xrightarrow{a} <\surd,\tfrac{1}{n}>,\ n = 1,\ldots,i+1$$

$$<q,0> \xrightarrow{\tau} <q_0,0>$$
$$<q_i,0> \xrightarrow{\tau} <q_{i+1},0>$$
$$<q_{i+1},0> \xrightarrow{\tau} <q_i,0>$$
$$<q_i,u> \overset{v-u}{\longmapsto} <q_i,v>,\ 0 \leq u < v \leq 1$$
$$<q_i,\tfrac{1}{n}> \xrightarrow{a} <\surd,\tfrac{1}{n}>,\ n = 1,\ldots,i+1$$

$$<r,0> \xrightarrow{\tau} <r_0,0>$$
$$<r_i,0> \xrightarrow{\tau} <r_{i+1},0>$$
$$<r_{i+1},0> \xrightarrow{\tau} <r_i,0>$$
$$<r_i,u> \overset{v-u}{\longmapsto} <r_i,v>,\ \tfrac{1}{i+2} \leq u < v \leq 1$$
$$<r_i,\tfrac{1}{n}> \xrightarrow{a} <\surd,\tfrac{1}{n}>,\ n = 1,\ldots,i+1$$
$$<r_0,0> \xrightarrow{\tau} <r_\infty,0>$$
$$<r_\infty,0> \xrightarrow{\tau} <r_0,0>$$
$$<r_\infty,u> \overset{v-u}{\longmapsto} <r_\infty,v>,\ 0 \leq u < v \leq 1$$
$$<r_\infty,\tfrac{1}{n}> \xrightarrow{a} <\surd,\tfrac{1}{n}>,\ n \in \mathbb{N}$$

$<p,0> \leftrightarrow_{tb}^{BM} <q,0>$, since $<p,w>\ B\ <q,w>$ for $w \geq 0$, $<p_i,w>\ B\ <q_i,w>$ for $w \leq \frac{1}{i+2}$, and $<p'_i,w>\ B\ <q_i,w>$ for $w > 0$ (for $i \geq 0$) is a branching tail bisimulation.

Moreover, $<q,0> \leftrightarrow_{tb}^{BM} <r,0>$, since $<q,w>\ B\ <r,w>$ for $w \geq 0$, $<q_i,w>\ B\ <r_i,w>$ for $w \geq 0$, $<q_i,0>\ B\ <r_j,0>$, and $<q_i,w>\ B\ <r_\infty,w>$ for $w = 0 \vee w > \frac{1}{i+2}$ (for $i,j \geq 0$) is a branching tail bisimulation.

However, $<p,0> \not\leftrightarrow_{tb}^{BM} <r,0>$, since p cannot simulate r. This is due to the fact that none of the p_i can simulate r_∞. Namely, r_∞ can idle until time 1. p_i can only simulate this by executing a τ at time $\frac{1}{i+2}$, but the resulting process $<p'_i,\frac{1}{i+2}>$ is not timed branching bisimilar to $<r_\infty,\frac{1}{i+2}>$, since only the latter can execute action a at time $\frac{1}{i+2}$.

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