Is Timed Branching Bisimilarity an Equivalence Indeed?

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Abstract. We show that timed branching bisimilarity as defined by van der Zwaag [14] and Baeten & Middelburg [2] is not an equivalence relation, in case of a dense time domain. We propose an adaptation based on van der Zwaag's definition, and prove that the resulting timed branching bisimilarity is an equivalence indeed. Furthermore, we prove that in case of a discrete time domain, van der Zwaag's definition and our adaptation coincide.

1 Introduction

Branching bisimilarity [6, 7] is a widely used concurrency semantics for process algebras that include the silent step τ. Two processes are branching bisimilar if they can be related by some branching bisimulation relation. See [5] for a clear account on the strong points of branching bisimilarity.

Over the years, process algebras such as CCS, CSP and ACP have been extended with a notion of time. As a result, the concurrency semantics underlying these process algebras have been adapted to cope with the presence of time. Klusener [11–13] was the first to extend the notion of a branching bisimulation relation to a setting with time. The main complication is that while a process can let time pass without performing an action, such idling may mean that certain behavioural options in the future are being discarded. Klusener pioneered how this aspect of timed processes can be taken into account in a branching bisimulation context. Based on his work, van der Zwaag [14, 15] and Baeten & Middelburg [2] proposed new notions of a timed branching bisimulation relation.

A key property for a semantics is that it is an equivalence. In general, for concurrency semantics in the presence of τ, reflexivity and symmetry are easy to see, but transitivity is much more difficult. In particular, the transitivity proof for branching bisimilarity in [6] turned out to be flawed, because the transitive closure of two branching bisimulation relations need not be a branching bisimulation relation. Basten [3] pointed out this flaw, and proposed a new transitivity proof for branching bisimilarity, based on the notion of a *semi*-branching bisimulation relation. Such relations are preserved under transitive closure, and the notions of branching bisimilarity and semi-branching bisimilarity coincide.

In a setting with time, proving equivalence of a concurrency semantics becomes even more complicated, compared to the untimed case. Still, equivalence properties for timed semantics are often claimed, but hardly ever proved. In [13–15, 2], equivalence properties are claimed without an explicit proof, although in all cases it is stated that such proofs do exist.

In the current paper, we study in how far for the notion of timed branching bisimilarity of van der Zwaag constitutes an equivalence relation. We give a counter-example to show that in case of a dense time domain, his notion is not transitive. We proceed to present a stronger version of van der Zwaag's definition (stronger in the sense that it relates fewer processes), and prove that this adapted notion does constitute an equivalence relation, even when the time domain is dense. Our proof follows the approach of Basten. Next, we show that in case of a discrete time domain, van der Zwaag's notion of timed branching bisimilarity and our new notion coincide. So in particular, in case of a discrete time domain, van der Zwaag's notion does constitute an equivalence relation.

In the appendix we show that our counter-example for transitivity also applies to the notion of timed branching bisimilarity by Baeten & Middelburg in case of a dense time domain; see [2, Section 6.4.1]. So that notion does not constitute an equivalence relation as well.

This paper is organized as follows. Section 2 contains the preliminaries. Section 3 features a counter-example to show that the notion of timed branching bisimilarity by van der Zwaag is not an equivalence relation in case of a dense time domain. A new definition of timed branching bisimulation is proposed in Section 4, and we prove that our notion of timed branching bisimilarity is an equivalence indeed. In Section 5 we prove that in case of a discrete time domain, our definition and van der Zwaag's definition of timed branching bisimilarity coincide. Section 6 gives suggestions for future work. In the appendix, we show that our counter-example for transitivity also applies to the notion of timed branching bisimilarity by Baeten & Middelburg [2].

2 Timed labelled transition systems

Let Act be a nonempty set of visible actions, and τ a special action to represent internal events, with $\tau \notin Act$. We use Act_τ to denote $Act \cup \{\tau\}$.

The time domain $Time$ is a totally ordered set with a least element 0. We say that $Time$ is *discrete* if for each pair $u, v \in Time$ there are only finitely many $w \in Time$ such that $u < w < v$.

Definition 1 ([14]). *A* timed labelled transition system *(TLTS) [8] is a tuple* (S, T, U)*, where:*

1. *S is a set of states, including a special state $\surd$ to represent successful termination;*

2. $T \subseteq S \times Act_\tau \times Time \times S$ *is a set of transitions;*
3. $U \subseteq S \times Time$ *is a* delay relation*, which satisfies:*
   – *if* $T(s, \ell, u, r)$*, then* $U(s, u)$*;*
   – *if* $u < v$ *and* $U(s, v)$*, then* $U(s, u)$*.*

Transitions (s, ℓ, u, s') express that state s evolves into state s' by the execution of action ℓ at (absolute) time u. It is assumed that the execution of transitions does not consume any time. A transition (s, ℓ, u, s') is denoted by $s \xrightarrow{\ell}_u s'$. If $U(s, u)$, then state s can let time pass until time u; these predicates are used to express time deadlocks.

3 Van der Zwaag's timed branching bisimulation

Van Glabbeek and Weijland [7] introduced the notion of a *branching bisimulation* relation for untimed LTSs. Intuitively, a τ-transition $s \xrightarrow{\tau} s'$ is invisible if it does not lose possible behaviour (i.e., if s and s' can be related by a branching bisimulation relation). See [5] for a lucid exposition on the motivations behind the definition of a branching bisimulation relation.

The reflexive transitive closure of $\xrightarrow{\tau}$ is denoted by $\Rightarrow$.

Definition 2 ([7]). *Assume an untimed LTS. A symmetric binary relation* $B \subseteq S \times S$ *is a* branching bisimulation *if* sBt *implies:*

1. *if* $s \xrightarrow{\ell} s'$*, then*
   i *either* $\ell = \tau$ *and* $s'Bt$*,*
   ii *or* $t \Rightarrow \hat{t} \xrightarrow{\ell} t'$ *with* $sB\hat{t}$ *and* $s'Bt'$*;*
2. *if* $s\downarrow$*, then* $t \Rightarrow t'\downarrow$ *with* sBt'*.*

Two states s *and* t *are* branching bisimilar*, denoted by* $s \leftrightarrow_b t$*, if there is a branching bisimulation* B *with* sBt*.*

Van der Zwaag [14] defined a timed version of branching bisimulation, which takes into account time stamps of transitions and ultimate delays $U(s, u)$.

For $u \in Time$, the reflexive transitive closure of $\xrightarrow{\tau}_u$ is denoted by $\Rightarrow_u$.

Definition 3 ([14]). *Assume a TLTS* (S, T, U)*. A collection* B *of symmetric binary relations* $B_u \subseteq S \times S$ *for* $u \in Time$ *is a* timed branching bisimulation *if* sB_ut *implies:*

1. *if* $s \xrightarrow{\ell}_u s'$*, then*
   i *either* $\ell = \tau$ *and* $s'B_ut$*,*
   ii *or* $t \Rightarrow_u \hat{t} \xrightarrow{\ell}_u t'$ *with* $sB_u\hat{t}$ *and* $s'B_ut'$*;*
2. *if* $s\downarrow$*, then* $t \Rightarrow_u t'\downarrow$ *with* sB_ut'*;*
3. *if* $u < v$ *and* $U(s, v)$*, then for some* $n > 0$ *there are* $t_0, \ldots, t_n \in S$ *with* $t = t_0$ *and* $U(t_n, v)$*, and* $u_0 < \cdots < u_n \in Time$ *with* $u = u_0$ *and* $v = u_n$*, such that for* $i < n$*,* $t_i \Rightarrow_{u_i} t_{i+1}$*,* $sB_{u_i}t_{i+1}$ *and* $sB_{u_{i+1}}t_{i+1}$*.*

Two states s and t are timed branching bisimilar at u *if there is a timed branching bisimulation* B *with* sB_ut. *States* s *and* t *are* timed branching bisimilar*, denoted by* $s \underline{\leftrightarrow}_{tb}^{Z} t$,[4] *if they are timed branching bisimilar at all* $u \in \mathit{Time}$.

Transitions can be executed at the same time consecutively. By the first clause in Definition 3, the behavior of a state at some point in time is treated like untimed behavior. The second clause deals with successful termination.[5] By the last clause, time passing in a state s is matched by a related state t with a "τ-path" where all intermediate states are related to s at times when a τ-transition is performed.

In the following examples, $\mathbb{Z}_{\geq 0} \subseteq \mathit{Time}$.

Example 1. Consider the following two TLTSs: $s_0 \xrightarrow{a}_2 s_1 \xrightarrow{b}_1 s_2$ and $t_0 \xrightarrow{a}_2 t_1$. We have $s_0 \underline{\leftrightarrow}_{tb}^{Z} t_0$, since $s_0 B_w t_0$ for $w \geq 0$, $s_1 B_w t_1$ for $w > 1$, and $s_2 B_w t_1$ for $w \geq 0$ is a timed branching bisimulation.

Example 2. Consider the following two TLTSs: $s_0 \xrightarrow{a}_1 s_1 \xrightarrow{\tau}_2 s_2 \xrightarrow{b}_3 s_3$ and $t_0 \xrightarrow{a}_1 t_1 \xrightarrow{b}_3 t_2$. We have $s_0 \underline{\leftrightarrow}_{tb}^{Z} t_0$, since $s_0 B_w t_0$ for $w \geq 0$, $s_1 B_w t_1$ for $w \leq 2$, $s_2 B_w t_1$ for $w \geq 0$, and $s_3 B_w t_2$ for $w \geq 0$ is a timed branching bisimulation.

Example 3. Consider the following two TLTSs: $s_0 \xrightarrow{a}_u s_1 \xrightarrow{\tau}_v s_2 \downarrow$ and $t_0 \xrightarrow{a}_u t_1 \downarrow$. If $u = v$, we have $s_0 \underline{\leftrightarrow}_{tb}^{Z} t_0$, since $s_0 B_w t_0$ for $w \geq 0$, $s_1 B_u t_1$, and $s_2 B_w t_1$ for $w \geq 0$ is a timed branching bisimulation. If $u \neq v$, we have $s_0 \not\underline{\leftrightarrow}_{tb}^{Z} t_0$, because s_1 and t_1 are not timed branching bisimilar at time u; namely, t_1 has a successful termination, and s_1 cannot simulate this at time u, as it cannot do a τ-transition at time u.

Example 4. Consider the following two TLTSs: $s_0 \xrightarrow{\tau}_u s_1 \xrightarrow{a}_v s_2 \downarrow$ and $t_0 \xrightarrow{a}_v t_1 \downarrow$. If $u = v$, we have $s_0 \underline{\leftrightarrow}_{tb}^{Z} t_0$, since $s_0 B_w t_0$ for $w \geq 0$, $s_1 B_w t_0$ for $w \geq 0$, and $s_2 B_w t_1$ for $w \geq 0$ is a timed branching bisimulation. If $u \neq v$, we have $s_0 \not\underline{\leftrightarrow}_{tb}^{Z} t_0$, because s_0 and t_0 are not timed branching bisimilar at time $\frac{u+v}{2}$.[6]

Van der Zwaag [14, 15] wrote about his definition: "It is straightforward to verify that branching bisimilarity is an equivalence relation." However, we found that in general this is not the case. A counter-example is presented below. Note that it uses a non-discrete time domain.

Example 5. Let p, q, and r defined as in Figures 1, 2 and 3, with $\mathit{Time} = \mathbb{Q}_{\geq 0}$. We depict $s \xrightarrow{a}_u s'$ as $s \xrightarrow{a(u)} s'$.

$p \underline{\leftrightarrow}_{tb}^{Z} q$, since pB_wq for $w \geq 0$, $p_iB_wq_i$ for $w \leq \frac{1}{i+2}$, and $p'_iB_wq_i$ for $w > 0$ (for $i \geq 0$) is a timed branching bisimulation.

[4] The superscript Z refers to van der Zwaag, to distinguish it from the adaptation of his definition of timed branching bisimulation that we will define later.

[5] Van der Zwaag does not take into account successful termination, so the second clause is missing in his definition.

[6] $s_0 \underline{\leftrightarrow}_{tb} t_0$ would hold for $u < v$ if in Definition 3 we would require that they are timed branching bisimilar at 0 (instead of at all $u \in \mathit{Time}$).

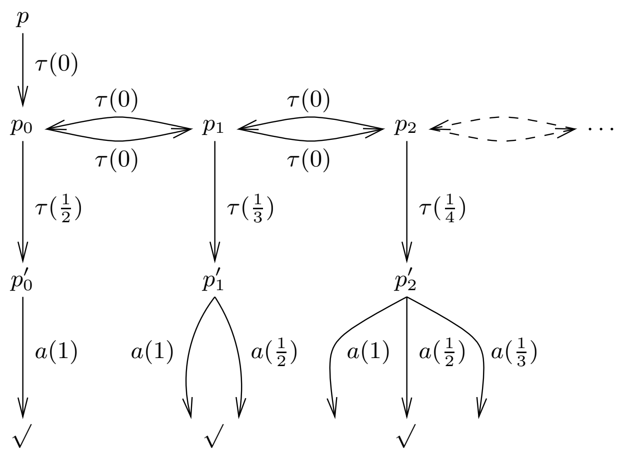

Fig. 1. A timed process p

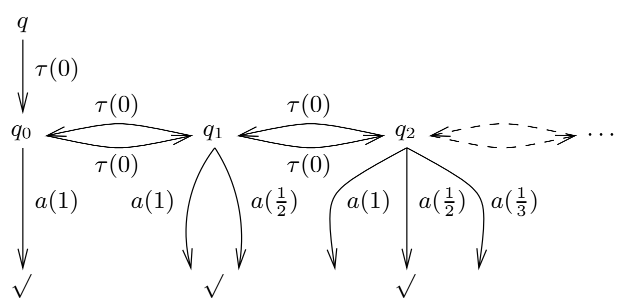

Fig. 2. A timed process q

Moreover, $q \underline{\leftrightarrow}^Z_{tb}\ r$, since qB_wr for $w \geq 0$, $q_iB_wr_i$ for $w \geq 0$, $q_iB_0r_j$, and $q_iB_wr_\infty$ for $w = 0 \vee w > \frac{1}{i+2}$ (for $i, j \geq 0$) is a timed branching bisimulation. (Note that q_i and r_∞ are *not* timed branching bisimilar in the time interval $\langle 0, \frac{1}{i+2}]$.)

However, $p \not\underline{\leftrightarrow}^Z_{tb}\ r$, due to the fact that none of the p_i can simulate r_∞. Namely, r_∞ can idle until time 1; p_i can only simulate this by executing a τ at time $\frac{1}{i+2}$, but the resulting process $\sum_{n=1}^{i+1} a(\frac{1}{n})$ is not timed branching bisimilar to r_∞ at time $\frac{1}{i+2}$, since only the latter can execute action a at time $\frac{1}{i+2}$.

4 A strengthened timed branching bisimulation

In this section, we propose a way to fix the definition of van der Zwaag (see Definition 3). Our adaptation requires the *stuttering property* [7] (see Definition 6) at all time intervals. That is, in the last clause of Definition 3, we require that

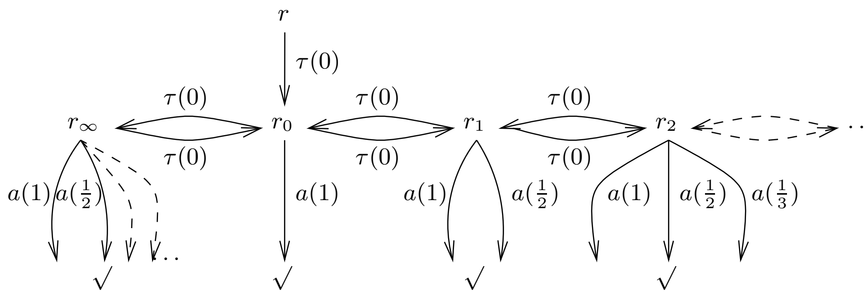

Fig. 3. A timed process r

sB_wt_{i+1} for $u_i \leq w \leq u_{i+1}$. Hence, we achieve a stronger version of van der Zwaag's definition. We prove that this new notion of timed branching bisimilarity is an equivalence relation.

4.1 Timed branching bisimulation

Definition 4. *Assume a TLTS* (S, T, U). *A collection* B *of binary relations* $B_u \subseteq S \times S$ *for* $u \in Time$ *is a* timed branching bisimulation *if* sB_ut *implies:*

1. *if* $s \xrightarrow{\ell}_u s'$, *then*
   i either $\ell = \tau$ *and* $s'B_ut$,
   ii or $t \Rightarrow_u \hat{t} \xrightarrow{\ell}_u t'$ *with* $sB_u\hat{t}$ *and* $s'B_ut'$;
2. *if* $t \xrightarrow{\ell}_u t'$, *then*
   i either $\ell = \tau$ *and* sB_ut',
   ii or $s \Rightarrow_u \hat{s} \xrightarrow{\ell}_u s'$ *with* $\hat{s}B_ut$ *and* $s'B_ut'$;
3. *if* $s\downarrow$, *then* $t \Rightarrow_u t'\downarrow$ *with* sB_ut';
4. *if* $t\downarrow$, *then* $s \Rightarrow_u s'\downarrow$ *with* $s'B_ut$;
5. *if* $u < v$ *and* $U(s, v)$, *then for some* $n > 0$ *there are* $t_0, \ldots, t_n \in S$ *with* $t = t_0$ *and* $U(t_n, v)$, *and* $u_0 < \cdots < u_n \in Time$ *with* $u = u_0$ *and* $v = u_n$, *such that for* $i < n$, $t_i \Rightarrow_{u_i} t_{i+1}$ *and* sB_wt_{i+1} *for* $u_i \leq w \leq u_{i+1}$;
6. *if* $u < v$ *and* $U(t, v)$, *then for some* $n > 0$ *there are* $s_0, \ldots, s_n \in S$ *with* $s = s_0$ *and* $U(s_n, v)$, *and* $u_0 < \cdots < u_n \in Time$ *with* $u = u_0$ *and* $v = u_n$, *such that for* $i < n$, $s_i \Rightarrow_{u_i} s_{i+1}$ *and* $s_{i+1}B_wt$ *for* $u_i \leq w \leq u_{i+1}$.

Two states s *and* t *are* timed branching bisimilar at u *if there is a timed branching bisimulation* B *with* sB_ut. *States* s *and* t *are* timed branching bisimilar, *denoted by* $s \underline{\leftrightarrow}_{tb}\, t$, *if they are timed branching bisimilar at all* $u \in Time$.

It is not hard to see that the union of timed branching bisimulations is again a timed branching bisimulation.

Note that states q and r from Example 5 are not timed branching bisimilar according to Definition 4. Namely, none of the q_i can simulate r_∞ in the time interval $\langle 0, \frac{1}{i+2}]$, so that the stuttering property is violated.

Starting from this point, we focus on timed branching bisimulation as defined in Definition 4. We did not define this new notion of timed branching bisimulation as a symmetric relation (like in Definition 3), in view of the equivalence proof that we are going to present. Namely, in general the relation composition of two symmetric relations is not symmetric. Clearly any symmetric timed branching bisimulation is a timed branching bisimulation. Furthermore, it follows from Definition 4 that the inverse of a timed branching bisimulation is again a timed branching bisimulation, so the union of a timed branching bisimulation and its inverse is a symmetric timed branching bisimulation. Hence, Definition 4 and the definition of timed branching bisimulation as a symmetric relation give rise to the same notion.

4.2 Timed semi-branching bisimulation

Basten [3] showed that the relation composition of two (untimed) branching bisimulations is not necessarily again a branching bisimulation. Figure 4 illustrates an example, showing that the relation composition of two timed branching bisimulations is not always a timed branching bisimulation. It is a slightly simplified version of an example from [3], here applied at time 0. Clearly, B and D are timed branching bisimulations. However, $B \circ D$ is not, and the problem arises at the transition $r_0 \xrightarrow{\tau}_0 r_1$. According to case 1 of Definition 3, since $r_0\,(B \circ D)\,t_0$, either $r_1\,(B \circ D)\,t_0$, or $r_0\,(B \circ D)\,t_1$ and $r_1\,(B \circ D)\,t_2$, must hold. But neither of these cases hold, so $B \circ D$ is not a timed branching bisimulation.

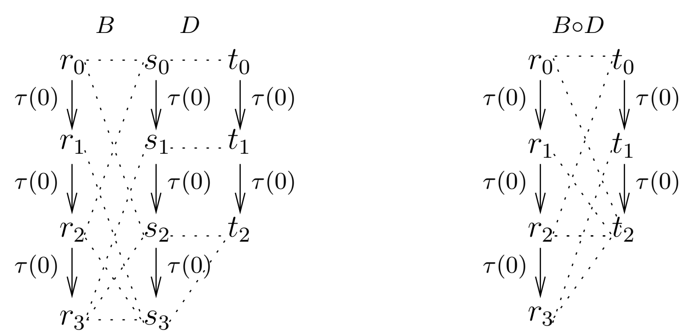

Fig. 4. Composition does not preserve timed branching bisimulation

Semi-branching bisimulation [7] relaxes case 1i of Definition 2: if $s \xrightarrow{\tau} s'$, then it is allowed that $t \Rightarrow t'$ with sBt' and $s'Bt'$. Basten proved that the relation composition of two semi-branching bisimulations is again a semi-branching bisimulation. It is easy to see that semi-branching bisimilarity is reflexive and symmetric. Hence, semi-branching bisimilarity is an equivalence relation. Then he proved that semi-branching bisimilarity and branching bisimilarity coincide, that means two states in an (untimed) LTS are related by a branching bisimulation relation if and only if they are related by a semi-branching bisimulation relation. We mimic the approach in [3] to prove that timed branching bisimilarity is an equivalence relation.

Definition 5. *Assume a TLTS (S, T, U). A collection B of binary relations $B_u \subseteq S \times \mathit{Time} \times S$ for $u \in \mathit{Time}$ is a* timed semi-branching bisimulation *if sB_ut implies:*

1. *if $s \xrightarrow{\ell}_u s'$, then*
   i *either $\ell = \tau$ and $t \Rightarrow_u t'$ with sB_ut' and $s'B_ut'$,*
   ii *or $t \Rightarrow_u \hat{t} \xrightarrow{\ell}_u t'$ with $sB_u\hat{t}$ and $s'B_ut'$;*
2. *if $t \xrightarrow{\ell}_u t'$, then*
   i *either $\ell = \tau$ and $s \Rightarrow_u s'$ with $s'B_ut$ and $s'B_ut'$,*
   ii *or $s \Rightarrow_u \hat{s} \xrightarrow{\ell}_u s'$ with $\hat{s}B_ut$ and $s'B_ut'$;*
3. *if $s\downarrow$, then $t \Rightarrow_u t'\downarrow$ with sB_ut';*
4. *if $t\downarrow$, then $s \Rightarrow_u s'\downarrow$ with $s'B_ut$;*
5. *if $u < v$ and $U(s, v)$, then for some $n > 0$ there are $t_0, \ldots, t_n \in S$ with $t = t_0$ and $U(t_n, v)$, and $u_0 < \cdots < u_n \in \mathit{Time}$ with $u = u_0$ and $v = u_n$, such that for $i < n$, $t_i \Rightarrow_{u_i} t_{i+1}$ and sB_wt_{i+1} for $u_i \leq w \leq u_{i+1}$;*
6. *if $u < v$ and $U(t, v)$, then for some $n > 0$ there are $s_0, \ldots, s_n \in S$ with $s = s_0$ and $U(s_n, v)$, and $u_0 < \cdots < u_n \in \mathit{Time}$ with $u = u_0$ and $v = u_n$, such that for $i < n$, $s_i \Rightarrow_{u_i} s_{i+1}$ and $s_{i+1}B_wt$ for $u_i \leq w \leq u_{i+1}$.*

Two states s and t are timed semi-branching bisimilar at u *if there is a timed semi-branching bisimulation B with sB_ut. States s and t are* timed semi-branching bisimilar, *denoted by $s \underline{\leftrightarrow}_{tsb} t$, if they are timed semi-branching bisimilar at all $u \in \mathit{Time}$.*

It is not hard to see that the union of timed semi-branching bisimulations is again a timed semi-branching bisimulation. Furthermore, any timed branching bisimulation is a timed semi-branching bisimulation.

Definition 6 ([7]). *A timed semi-branching bisimulation B is said to satisfy the* stuttering property *if:*

1. *sB_ut, $s'B_ut$ and $s \xrightarrow{\tau}_u s_1 \xrightarrow{\tau}_u \cdots \xrightarrow{\tau}_u s_n \xrightarrow{\tau}_u s'$ implies that s_iB_ut for $1 \leq i \leq n$;*
2. *sB_ut, sB_ut' and $t \xrightarrow{\tau}_u t_1 \xrightarrow{\tau}_u \cdots \xrightarrow{\tau}_u t_n \xrightarrow{\tau}_u t'$ implies that sB_ut_i for $1 \leq i \leq n$.*

Lemma 1. *Any timed semi-branching bisimulation satisfying the stuttering property is a timed branching bisimulation.*

Proof. Let B be a timed semi-branching bisimulation that satisfies the stuttering property. We prove that B is a timed branching bisimulation.

Let sB_ut. We only consider case 1i of Definition 5, because cases 1ii, 2ii and 3-6 are the same for both timed semi-branching and branching bisimulation. Moreover, case 2i can be dealt with in a similar way as case 1i. So let $s \xrightarrow{\tau}_u s'$ and $t \Rightarrow_u t'$ with sB_ut' and $s'B_ut'$. We distinguish two cases.

1. $t = t'$. Then $s'B_u t$, which agrees with case 1i of Definition 4.
2. $t \neq t'$. Then $t \Rightarrow_u t'' \xrightarrow{\tau}_u t'$. Since B satisfies the stuttering property, $sB_u t''$. This agrees with case 1ii of Definition 4. □

4.3 Timed branching bisimilarity is an equivalence.

Our equivalence proof consists of the following main steps:

1. We first prove that the relation composition of two timed semi-branching bisimulation relations is again a semi-branching bisimulation relation (Proposition 1).
2. Then we prove that timed semi-branching bisimilarity is an equivalence relation (Theorem 1).
3. Finally, we prove that the largest timed semi-branching bisimulation satisfies the stuttering property (Proposition 2).

According to Lemma 1, any timed semi-branching bisimulation satisfying the stuttering property is a timed branching bisimulation. So by the 3rd point, two states are related by a timed branching bisimulation if and only if they are related by a timed semi-branching bisimulation.

Lemma 2. *Let B be a timed semi-branching bisimulation, and $sB_u t$.*

1. $s \Rightarrow_u s' \implies (\exists t' \in S : t \Rightarrow_u t' \wedge s'B_u t')$;
2. $t \Rightarrow_u t' \implies (\exists s' \in S : s \Rightarrow_u s' \wedge t'B_u s')$.

Proof. We prove the first part, by induction on the number of τ-transitions at u from s to s'.

1. *Base case:* The number of τ-transitions at u from s to s' is zero. Then $s = s'$. Take $t' = t$. Clearly $t \Rightarrow_u t'$ and $s'B_u t'$.
2. *Inductive case:* $s \Rightarrow_u s'$ consists of $n \geq 1$ τ-transitions at u. Then there exists an $s'' \in S$ such that $s \Rightarrow_u s''$ in $n - 1$ τ-transitions at u, and $s'' \xrightarrow{\tau}_u s'$. By the induction hypothesis, $t \Rightarrow_u t''$ with $s''B_u t''$. Since $s'' \xrightarrow{\tau}_u s'$ and B is a timed semi-branching bisimulation:
   - either $t'' \Rightarrow_u t'$ and $s''B_u t'$ and $s'B_u t'$;
   - or $t'' \Rightarrow_u \hat{t} \xrightarrow{\tau}_u t'$ with $s''B_u \hat{t}$ and $s'B_u t'$.

   In both cases $t \Rightarrow_u t'$ with $s'B_u t'$.

The proof of the second part is similar. □

Proposition 1. *The relation composition of two timed semi-branching bisimulations is again a timed semi-branching bisimulation.*

Proof. Let B and D be timed semi-branching bisimulations. We prove that the composition of B and D (or better, the compositions of B_u and D_u for $u \in \mathit{Time}$) is a timed semi-branching bisimulation. Suppose that $rB_u sD_u t$ for $r, s, t \in S$. We check that the conditions of Definition 5 are satisfied with respect to the pair r, t. We distinguish four cases.

1. $r \xrightarrow{\tau}_u r'$ and $s \Rightarrow_u s'$ with rB_us' and $r'B_us'$. Since sD_ut and $s \Rightarrow_u s'$, Lemma 2 yields that $t \Rightarrow_u t'$ with $s'D_ut'$. Hence, $rB_us'D_ut'$ and $r'B_us'D_ut'$.
2. $r \xrightarrow{\ell}_u r'$ and $s \Rightarrow_u s'' \xrightarrow{\ell}_u s'$ with rB_us'' and $r'B_us'$. Since sD_ut and $s \Rightarrow_u s''$, Lemma 2 yields that $t \Rightarrow_u t''$ with $s''D_ut''$. Since $s'' \xrightarrow{\ell}_u s'$ and $s''D_ut''$:
   - Either $\ell = \tau$ and $t'' \Rightarrow_u t'$ with $s''D_ut'$ and $s'D_ut'$. Then $t \Rightarrow_u t'$ with $rB_us''D_ut'$ and $r'B_us'D_ut'$.
   - Or $t'' \Rightarrow_u t''' \xrightarrow{\ell}_u t'$ with $s''D_ut'''$ and $s'D_ut'$. Then $t \Rightarrow_u t''' \xrightarrow{\ell}_u t'$ with $rB_us''D_ut'''$ and $r'B_us'D_ut'$.
3. $r\downarrow$. Since rB_us, $s \Rightarrow_u s' \downarrow$ with rB_us'. Since sD_ut and $s \Rightarrow_u s'$, Lemma 2 yields that $t \Rightarrow_u t''$ with $s'D_ut''$. Since $s' \downarrow$ and $s'D_ut''$, $t'' \Rightarrow_u t' \downarrow$ with $s'D_ut'$. Hence, $t \Rightarrow_u t' \downarrow$ with $rB_us'D_ut'$.
4. $u < v$ and $U(r, v)$. Since rB_us, for some $n > 0$ there are $s_0, \ldots, s_n \in S$ with $s = s_0$ and $U(s_n, v)$, and $u_0 < \cdots < u_n \in \mathit{Time}$ with $u = u_0$ and $v = u_n$, such that $s_i \Rightarrow_{u_i} s_{i+1}$ and rB_ws_{i+1} for $u_i \leq w \leq u_{i+1}$ and $i < n$.
   For $i \leq n$ we show that for some $m_i > 0$ there are $t^i_0, \ldots, t^i_{m_i} \in S$ with $t = t^0_0$ and $U(t^n_{m_n}, v)$, and $v^i_0 \leq \cdots \leq v^i_{m_i} \in \mathit{Time}$ with (A_i) $u_{i-1} = v^i_0$ (if $i > 0$) and (B_i) $u_i = v^i_{m_i}$, such that:
   (C_i) $t^i_j \Rightarrow_{v^i_j} t^i_{j+1}$ for $j < m_i$;
   (D_i) $t^{i-1}_{m_{i-1}} \Rightarrow_{u_{i-1}} t^i_0$ (if $i > 0$);
   (E_i) $s_iD_{u_{i-1}}t^i_0$ (if $i > 0$);
   (F_i) $s_iD_wt^i_{j+1}$ for $v^i_j \leq w \leq v^i_{j+1}$ and $j < m_i$.
   We apply induction with respect to i.
   - *Base case:* $i = 0$.
     Let $m_0 = 1$, $t^0_0 = t^0_1 = t$ and $v^0_0 = v^0_1 = u_0$. Note that B_0, C_0 and F_0 hold.
   - *Inductive case:* $0 < i \leq n$.
     Suppose that $m_k, t^k_0, \ldots, t^k_{m_k}, v^k_0, \ldots, v^k_{m_k}$ have been defined for $0 \leq k < i$. Moreover, suppose that B_k, C_k and F_k hold for $0 \leq k < i$, and that A_k, D_k and E_k hold for $0 < k < i$.
     F_{i-1} for $j = m_{i-1} - 1$ together with B_{i-1} yields $s_{i-1}D_{u_{i-1}}t^{i-1}_{m_{i-1}}$. Since $s_{i-1} \Rightarrow_{u_{i-1}} s_i$, Lemma 2 implies that $t^{i-1}_{m_{i-1}} \Rightarrow_{u_{i-1}} t'$ with $s_iD_{u_{i-1}}t'$. We define $t^i_0 = t'$ [then D_i and E_i hold] and $v^i_0 = u_{i-1}$ [then A_i holds]. $s_i \Rightarrow_{u_i} \cdots \Rightarrow_{u_{n-1}} s_n$ with $U(s_n, v)$ implies that $U(s_i, u_i)$. Since $s_iD_{u_{i-1}}t^i_0$, according to case 5 of Definition 5, for some $m_i > 0$ there are $t^i_1, \ldots, t^i_{m_i} \in S$ with $U(t^i_{m_i}, u_i)$, and $v^i_1 < \cdots < v^i_{m_i} \in \mathit{Time}$ with $v^i_0 < v^i_1$ and $u_i = v^i_{m_i}$ [then B_i holds], such that for $j < m_i$, $t^i_j \Rightarrow_{v^i_j} t^i_{j+1}$ [then C_i holds] and $s_iD_wt^i_{j+1}$ for $v^i_j \leq w \leq v^i_{j+1}$ [then F_i holds].

   Concluding, for $i < n$, $rB_{u_i}s_{i+1}D_{u_i}t^{i+1}_0$ and $rB_ws_{i+1}D_wt^{i+1}_{j+1}$ for $v^{i+1}_j \leq w \leq v^{i+1}_{j+1}$ and $j < m_i$. Since $v^i_j \leq v^i_{j+1}$, $v^i_{m_i} = u_i = v^{i+1}_0$, $t = t^0_0$, $u = u_0 = v^0_0$, $t^i_j \Rightarrow_{v^i_j} t^i_{j+1}$, $t^i_{m_i} \Rightarrow_{u_i} t^{i+1}_0$, and $U(t^n_{m_n}, v)$, we are done.

So cases 1,3,5 of Definition 5 are satisfied. Similarly it can be checked that cases 2,4,6 are satisfied. So the composition of B and D is again a timed semi-branching bisimulation. □

Theorem 1. *Timed semi-branching bisimilarity, $\underline{\leftrightarrow}_{tsb}$, is an equivalence relation.*

Proof. Reflexivity: Obviously, the identity relation on S is a timed semi-branching bisimulation.
Symmetry: Let B a timed semi-branching bisimulation. Obviously, B^{-1} is also a timed semi-branching bisimulation.
Transitivity: This follows from Proposition 1. □

Proposition 2. *The largest timed semi-branching bisimulation satisfies the stuttering property.*

Proof. Let B be the largest timed semi-branching bisimulation on S. Let $s \xrightarrow{\tau}_u s_1 \xrightarrow{\tau}_u \cdots \xrightarrow{\tau}_u s_n \xrightarrow{\tau}_u s'$ with sB_ut and $s'B_ut$. We prove that $B' = B \cup \{(s_i, t) \mid 1 \leq i \leq n\}$ is a timed semi-branching bisimulation.

We check that all cases of Definition 5 are satisfied for the relations $s_iB'_ut$, for $1 \leq i \leq n$. First we check that the transitions of s_i are matched by t. Since $s \Rightarrow_u s_i$ and sB_ut, by Lemma 2 $t \Rightarrow_u t'$ with s_iB_ut'.

- If $s_i \xrightarrow{\ell}_u s''$, then it follows from s_iB_ut' that:
  - Either $\ell = \tau$ and $t' \Rightarrow_u t''$ with s_iB_ut'' and $s''B_ut''$. Since $t \Rightarrow_u t' \Rightarrow_u t''$, this agrees with case 1i of Definition 5.
  - Or $t' \Rightarrow_u t''' \xrightarrow{\ell}_u t''$ with s_iB_ut''' and $s''B_ut''$. Since $t \Rightarrow_u t' \Rightarrow_u t'''$, this agrees with case 1ii of Definition 5.
- If $s_i \downarrow$, then it follows from s_iB_ut' that $t' \Rightarrow_u t'' \downarrow$ with s_iB_ut''. Since $t \Rightarrow_u t' \Rightarrow_u t''$, this agrees with case 3 of Definition 5.
- If $u < v$ and $U(s_i, v)$, then it follows from s_iB_ut' that for some $n > 0$ there are $t_0, \ldots, t_n \in S$ with $t' = t_0$ and $U(t_n, v)$, and $u_0 < \cdots < u_n \in \mathit{Time}$ with $u = u_0$ and $v = u_n$, such that for $i < n$, $t_i \Rightarrow_{u_i} t_{i+1}$ and $s_iB_wt_i$ for $u_i \leq w \leq u_{i+1}$. Since $t \Rightarrow_u t' \Rightarrow_u t_1$, this agrees with case 5 of Definition 5.

Next we check that the transitions of t are matched by s_i.

- If $t \xrightarrow{\ell}_u t'$, then it follows from $s'B_ut$ that:
  - Either $\ell = \tau$ and $s' \Rightarrow_u s''$ with $s''B_ut$ and $s''B_ut'$. Since $s_i \Rightarrow_u s' \Rightarrow_u s''$, this agrees with case 2i of Definition 5.
  - Or $s' \Rightarrow_u s''' \xrightarrow{\ell}_u s''$ with $s'''B_ut$ and $s''B_ut'$. Since $s_i \Rightarrow_u s' \Rightarrow_u s'''$, this agrees with case 2ii of Definition 5.
- If $t \downarrow$, then it follows from $s'B_ut$ that $s' \Rightarrow_u s'' \downarrow$ with $s''B_ut$. Since $s_i \Rightarrow_u s' \Rightarrow_u s''$, this agrees with case 4 of Definition 5.
- If $u < v$ and $U(t, v)$, then it follows from $s'B_ut$ that for some $n > 0$ there are $s'_0, \ldots, s'_n \in S$ with $s' = s'_0$ and $U(s_n, v)$, and $u_0 < \cdots < u_n \in \mathit{Time}$ with $u = u_0$ and $v = u_n$, such that for $i < n$, $s'_i \Rightarrow_{u_i} s'_{i+1}$ and $s'_{i+1}B_wt$ for $u_i \leq w \leq u_{i+1}$. Since $s_i \Rightarrow_u s' \Rightarrow_u s'_1$, this agrees with case 6 of Definition 5.

Hence B' is a timed semi-branching bisimulation. Since B is the largest, and $B \subseteq B'$, we find that $B = B'$. So B satisfies the first requirement of Definition 6.

Since B is the largest timed semi-branching bisimulation and $\underline{\leftrightarrow}_{tsb}$ is an equivalence, B is symmetric. Then B also satisfies the second requirement of Definition 6. Hence B satisfies the stuttering property. $\square$

As a consequence, the largest timed semi-branching bisimulation is a timed branching bisimulation (by Lemma 1 and Proposition 2). Since any timed branching bisimulation is a timed semi-branching bisimulation, we have the following two corollaries.

Corollary 1. *Two states are related by a timed branching bisimulation if and only if they are related by a timed semi-branching bisimulation.*

Corollary 2. *Timed branching bisimilarity, $\underline{\leftrightarrow}_{tb}$, is an equivalence relation.*

We note that for each $u \in \mathit{Time}$, timed branching bisimilarity at time u is also an equivalence relation.

5 Discrete time domains

Theorem 2. *In case of a discrete time domain, $\underline{\leftrightarrow}_{tb}^{Z}$ and $\underline{\leftrightarrow}_{tb}$ coincide.*

Proof. Clearly $\underline{\leftrightarrow}_{tb} \subseteq \underline{\leftrightarrow}_{tb}^{Z}$. We prove that $\underline{\leftrightarrow}_{tb}^{Z} \subseteq \underline{\leftrightarrow}_{tb}$. Suppose B is a timed branching bisimulation relation according to Definition 3. We show that B is a timed branching bisimulation relation according to Definition 4. B satisfies cases 1-4 of Definition 4, since they coincide with cases 1-2 of Definition 3. We prove that case 5 of Definition 4 is satisfied.

Let sB_ut and $U(s,v)$ with $u < v$. Let $u_0 < \cdots < u_n \in \mathit{Time}$ with $u_0 = u$ and $u_n = v$, where $u_1, \ldots, u_{n-1}$ are all the elements from Time that are between u and v. (Here we use that Time is discrete.) We prove induction on n that there are $t_0, \ldots, t_n \in S$ with $t = t_0$ and $U(t_n, v)$, such that for $i < n$, $t_i \Rightarrow_{u_i} t_{i+1}$ and sB_wt_{i+1} for $u_i \leq w \leq u_{i+1}$.

- *Base case:* $n = 1$. By case 3 of Definition 3 there is a $t_1 \in S$ with $U(t_1, v)$, such that $t \Rightarrow_u t_1$, sB_ut_1 and sB_vt_1. Hence, sB_wt_1 for $u \leq w \leq v$.
- *Inductive case:* $n > 1$. Since $U(s,v)$, clearly also $U(s,u_1)$. By case 3 of Definition 3 there is a $t_1 \in S$ such that $t \Rightarrow_u t_1$, sB_ut_1 and $sB_{u_1}t_1$. Hence, sB_wt_1 for $u \leq w \leq u_1$. By induction, $sB_{u_1}t_1$ together with $U(s,v)$ implies that there are $t_2, \ldots, t_n \in S$ with $U(t_n, v)$, such that for $1 \leq i < n$, $t_i \Rightarrow_{u_i} t_{i+1}$, $sB_{u_i}t_{i+1}$ and $sB_{u_{i+1}}t_{i+1}$. Hence, sB_wt_{i+1} for $u_i \leq w \leq u_{i+1}$.

We conclude that case 5 of Definition 4 holds. Similarly it can be proved that B satisfies case 6 of Definition 4. Hence B is a timed branching bisimulation relation according to Definition 4. So $\underline{\leftrightarrow}_{tb}^{Z} \subseteq \underline{\leftrightarrow}_{tb}$. $\square$

6 Future work

We conclude the paper by pointing out some possible research directions for the future.

1. It is an interesting question whether a rooted version of timed branching bisimilarity is a congruence over a basic timed process algebra (such as Baeten and Bergstra's $\mathrm{BPA}^{ur}_{\rho\delta}$ [1], which is basic real time process algebra with time stamped urgent actions). Similar to equivalence, congruence properties for timed branching bisimilarity are often claimed, but hardly ever proved. We only know of one such congruence proof, in [13].
2. Van der Zwaag [14] extended the cones and foci verification method from Groote and Springintveld [9] to TLTSs. Fokkink and Pang [10] proposed an adapted version of this timed cones and foci method. Both papers take $\underline{\leftrightarrow}^Z_{tb}$ as a starting point. It should be investigated whether a timed cones and foci method can be formulated for $\underline{\leftrightarrow}_{tb}$ as defined in the current paper.
3. Van Glabbeek [4] presented a wide range of concurrency semantics for untimed processes with the silent step τ. It would be a challenge to try and formulate timed versions of these semantics, and prove equivalence and congruence properties for the resulting timed semantics.

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A Branching tail bisimulation

Baeten and Middelburg [2] defined the notion of *branching tail bisimulation*, which is closely related to van der Zwaag's definition of timed branching bisimulation. We show that in case of dense time, our counter-example (see Example 5) again shows that branching tail bisimilarity is not an equivalence relation.

In the absolute time setting of Baeten and Middelburg, states are of the form $<p,u>$ with p a process algebraic term and u a time stamp referring to the absolute time. They give operational semantics to their process algebras such that if $<p,u> \overset{v}{\longmapsto} <p,u{+}v>$ (where $\overset{v}{\longmapsto}$ for $v > 0$ denotes a time step of v time units), then $<p,u> \overset{w}{\longmapsto} <p,u{+}w>$ for $0 < w < v$; in our example this saturation with time steps will be mimicked. The relation $s \overset{u}{\longmapsto\!\!\!\twoheadrightarrow} s'$ is defined by: either $s \Rightarrow \hat{s} \overset{u}{\longmapsto} s'$, or $s \overset{v}{\longmapsto\!\!\!\twoheadrightarrow} \hat{s} \overset{w}{\longmapsto\!\!\!\twoheadrightarrow} s'$ with $v + w = u$.[7]

Branching tail bisimulation is defined as follows.[8]

Definition 7 ([2]). *Assume a TLTS in the style of Baeten and Middelburg. A symmetric binary relation $B \subseteq S \times S$ is a* branching tail bisimulation *if sBt implies:*

1. *if $s \xrightarrow{\ell} s'$, then*
   - *i either $\ell = \tau$ and $t \Rightarrow t'$ with sBt' and $s'Bt'$;*
   - *ii or $t \Rightarrow \hat{t} \xrightarrow{a} t'$ with $sB\hat{t}$ and $s'Bt'$;*
2. *if $s \xrightarrow{\ell} <\surd, u>$, then $t \Rightarrow t' \xrightarrow{\ell} <\surd, u>$ with sBt';*
3. *if $s \overset{u}{\longmapsto} s'$, then*
   - *i either $t \Rightarrow \hat{t} \overset{v}{\longmapsto} \hat{t}' \overset{w}{\longmapsto\!\!\!\twoheadrightarrow} t'$ with $v + w = u$, $sB\hat{t}$ and $s'Bt'$;*
   - *ii or $t \Rightarrow \hat{t} \overset{u}{\longmapsto} t'$ with $sB\hat{t}$ and $s'Bt'$.*

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[7] Baeten and Middelburg also have a deadlock predicate $\uparrow$, which we do not take into account here, as it does not play a role in our counter-example.

[8] Baeten and Middelburg define this notion in the setting with relative time, and remark that the adaptation of this definition to absolute time is straightforward. Here we present this straightforward adaptation.

Two states s and t are branching tail bisimilar*, written $s \underline{\leftrightarrow}_{tb}^{BM} t$, if there is a branching tail bisimulation B with sBt.*[9]

We proceed to transpose the TLTSs from Example 5 into the setting of Baeten and Middelburg. We now have the following transitions, for $i \geq 0$:

$<p,0> \xrightarrow{\tau} <p_0,0>$
$<p_i,0> \xrightarrow{\tau} <p_{i+1},0>$
$<p_{i+1},0> \xrightarrow{\tau} <p_i,0>$
$<p_i,u> \overset{v-u}{\longmapsto} <p_i,v>$, $0 \leq u < v \leq \frac{1}{i+2}$
$<p_i,\frac{1}{i+2}> \xrightarrow{\tau} <p_i',\frac{1}{i+2}>$
$<p_i',u> \overset{v-u}{\longmapsto} <p_i',v>$, $\frac{1}{i+2} \leq u < v \leq 1$
$<p_i',\frac{1}{n}> \xrightarrow{a} <\surd,\frac{1}{n}>$, $n = 1,\ldots,i+1$

$<q,0> \xrightarrow{\tau} <q_0,0>$
$<q_i,0> \xrightarrow{\tau} <q_{i+1},0>$
$<q_{i+1},0> \xrightarrow{\tau} <q_i,0>$
$<q_i,u> \overset{v-u}{\longmapsto} <q_i,v>$, $0 \leq u < v \leq 1$
$<q_i,\frac{1}{n}> \xrightarrow{a} <\surd,\frac{1}{n}>$, $n = 1,\ldots,i+1$

$<r,0> \xrightarrow{\tau} <r_0,0>$
$<r_i,0> \xrightarrow{\tau} <r_{i+1},0>$
$<r_{i+1},0> \xrightarrow{\tau} <r_i,0>$
$<r_i,u> \overset{v-u}{\longmapsto} <r_i,v>$, $\frac{1}{i+2} \leq u < v \leq 1$
$<r_i,\frac{1}{n}> \xrightarrow{a} <\surd,\frac{1}{n}>$, $n = 1,\ldots,i+1$
$<r_0,0> \xrightarrow{\tau} <r_\infty,0>$
$<r_\infty,0> \xrightarrow{\tau} <r_0,0>$
$<r_\infty,u> \overset{v-u}{\longmapsto} <r_\infty,v>$, $0 \leq u < v \leq 1$
$<r_\infty,\frac{1}{n}> \xrightarrow{a} <\surd,\frac{1}{n}>$, $n \in \mathbb{N}$

$<p,0> \underline{\leftrightarrow}_{tb}^{BM} <q,0>$, since $<p,w>B<q,w>$ for $w \geq 0$, $<p_i,w>B<q_i,w>$ for $w \leq \frac{1}{i+2}$, and $<p_i',w>B<q_i,w>$ for $w > 0$ (for $i \geq 0$) is a branching tail bisimulation.

Moreover, $<q,0> \underline{\leftrightarrow}_{tb}^{BM} <r,0>$, since $<q,w>B<r,w>$ for $w \geq 0$, $<q_i,w>B<r_i,w>$ for $w \geq 0$, $<q_i,0>B<r_j,0>$, and $<q_i,w>B<r_\infty,w>$ for $w = 0 \vee w > \frac{1}{i+2}$ (for $i,j \geq 0$) is a branching tail bisimulation.

However, $<p,0> \not\underline{\leftrightarrow}_{tb}^{BM} <r,0>$, since p cannot simulate r. This is due to the fact that none of the p_i can simulate r_∞. Namely, r_∞ can idle until time 1. p_i can only simulate this by executing a τ at time $\frac{1}{i+2}$, but the resulting process $<p_i',\frac{1}{i+2}>$ is not timed branching bisimilar to $<r_\infty,\frac{1}{i+2}>$, since only the latter can execute action a at time $\frac{1}{i+2}$.

[9] The superscript BM refers to Baeten and Middelburg, to distinguish it from the notion of timed branching bisimulation as defined in this paper.